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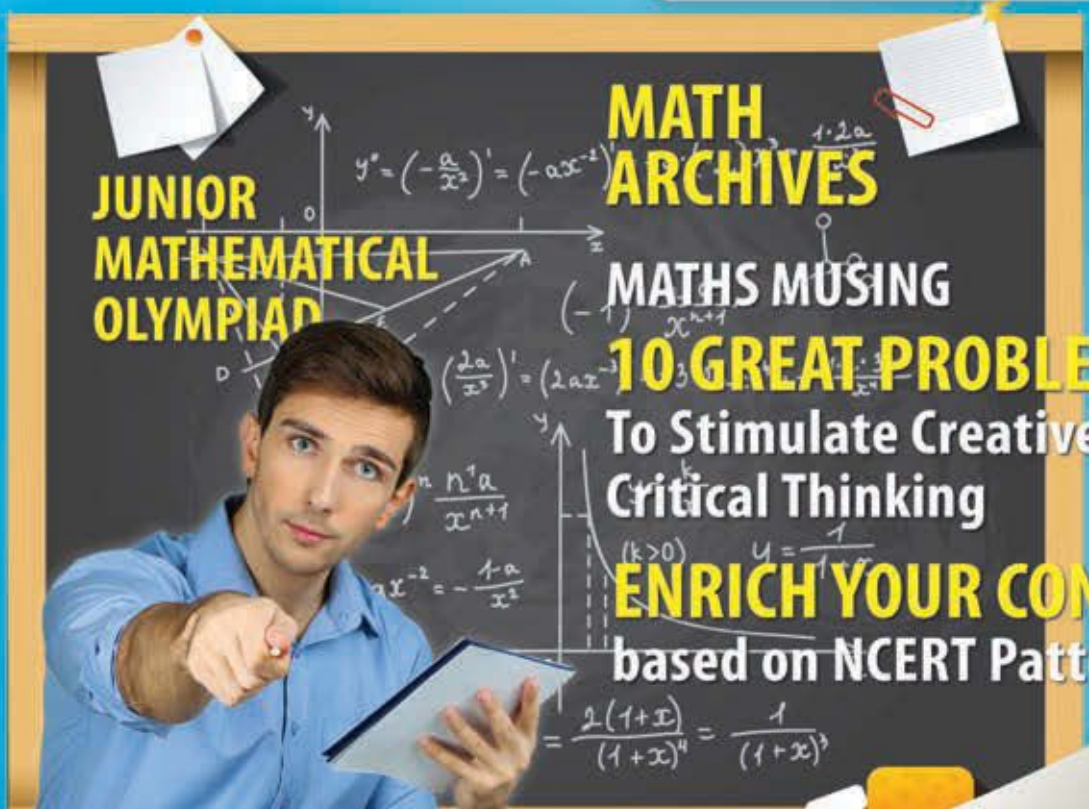

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Vol. XXXIII No. 1 January 2015

Corporate Office
Plot 99, Sector 44 Institutional Area, Gurgaon, (HR).
Tel : 0124-4951200
e-mail : info@mtg.in website : www.mtg.in

Regd. Office
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110 029.

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editorial

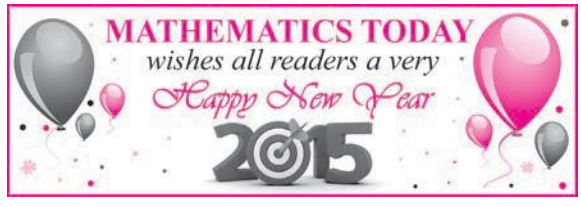
Maths and Science & Technology

Proficiency in Mathematics is always considered to be important in engineering. One does not know where suddenly one finds the need of mathematics till one is pushed to a corner. Earlier, students of agriculture were not very much bothered about mathematics. Yet there is no research without application of mathematics in one form or another.

How does one know whether a particular treatment is useful to get better cotton, jute, silk or even artificial fibres? First we have to know what is a good cotton. One studies all the physical properties such as fibre-length, strength, ability to form strong threads and so on. Changes due to various treatments have to be significant statistically.

The minimum number of experiments that are needed to determine a particular property – even energy levels of atoms, the wavelengths of the various lines of spectra, the calculation of energy levels are all determined by statistics. To determine the structure of molecules or crystals, group theory is the main tool. There are no hard-bound barriers dividing any two branches in science. To study in a systematic manner, we divide it into Maths, Physics, Chemistry, Agriculture or Industrial research. 'Science is a single field'.

Anil Ahlawat
Editor



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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material. During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 145

JEE MAIN

1. $0 < x < y < 2015$. The number of pairs of integers (x, y) such that A.M. of x and y exceeds their G.M. by 2 is

(a) 42 (b) 43 (c) 44 (d) 41

2. $0 < x < y$ are integers such that $\frac{1}{x}+\frac{1}{y}=\frac{1}{2015}$. The number of pairs (x, y) is

(a) 12 (b) 13 (c) 14 (d) 11

3. In a right angled triangle with integer sides, the inradius is 2015. The number of such triangles is

(a) 24 (b) 25 (c) 26 (d) 27

4. If $x + 4y + 9z = 305$, $4x + 9y + 16z = 310$, $9x + 16y + 25z = 880$, then $16x + 25y + 36z =$

(a) 2015 (b) 2016 (c) 2014 (d) 2010

5. If the quadratic equation $x^2 + ax + 6a = 0$ has only integer roots, then the number of values of a is

(a) 7 (b) 8 (c) 9 (d) 10

JEE ADVANCED

6. If $f(x)=\begin{cases}3x^2+12x-1, & -1\le x\le 2\\ 37-x, & 2<x\le 3\end{cases}$, then

(a) $f(x)$ increases in $[-1, 2]$
(b) $f(x)$ is continuous in $[-1, 3]$
(c) $f'(x)$ does not exist at $x = 2$
(d) $f(x)$ has maximum at $x = 2$

COMPREHENSION

Circles of radii 5, 5, 8 touch each other externally.

7. The radius of the circle which touches all the three circles externally is

(a) $\frac{5}{9}$ (b) $\frac{7}{9}$ (c) $\frac{8}{9}$ (d) $\frac{2}{3}$

8. The radius of the circle which touches all the three circles internally is

(a) 13 (b) $\frac{40}{3}$ (c) $\frac{41}{3}$ (d) 14

INTEGER MATCH

9. A fair coin is tossed 10 times. The probability that head never occur on consecutive tosses is $\frac{m}{n}$, in lowest terms, where m is

MATCHING LIST

10. Match the following :

Column-I		Column-II	
P.	$\int_{-1}^{1}\cos^{-1}x\,dx=$	1.	$-\frac{\pi}{4}$
Q.	$\int_{-1}^{1}\sin^{-1}x\,dx$	2.	0
R.	$\int_{-1}^{1}x\cos^{-1}x\,dx$	3.	$\frac{\pi}{4}$
S.	$\int_{-1}^{1}x\sin^{-1}x\,dx$	4.	π

	P	Q	R	S
(a)	3	4	2	1
(b)	1	3	4	2
(c)	2	1	3	4
(d)	4	2	1	3

See Solution set of Maths Musing 144 on page no. 29.

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

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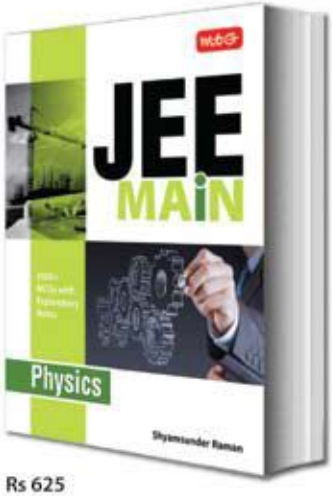

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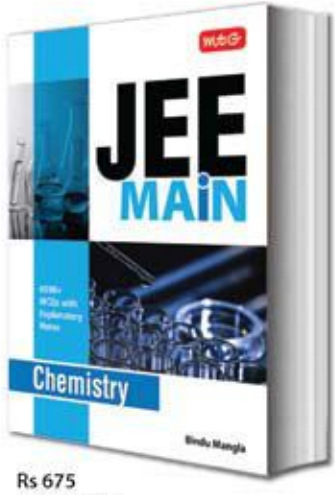

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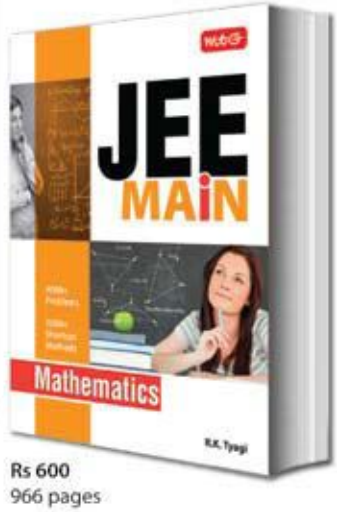

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PRACTICE PAPER 2015

JEE Main

Integral Calculus

1. $\lim_{n\to\infty} \sin\left(\frac{n}{n^2+1^2}\right)+\sin\left(\frac{n}{n^2+2^2}\right)+\ldots$
$\ldots.+\sin\left(\frac{n}{n^2+n^2}\right)=$

(a) π (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

2. Let f be defined by $f(x)=\sin\left(\frac{1}{x}\right),\ x\neq 0$
$= c,\ x = 0$
where $c \in [-1, 1]$. For which value of c does there exist an antiderivative of f ?

(a) $-1/2$ (b) $1/2$ (c) 0 (d) -1

3. The function $f(\lambda)=\int_0^{\pi/2}\frac{dx}{\sqrt{1-\lambda\cos^2 x}}$,
$\lambda \in (0, 1)$ is

(a) increasing (b) decreasing
(c) increasing on (0, 1/2) and decreasing on (1/2, 1)
(d) increasing on (1/2, 1) and decreasing on (0, 1/2)

4. $\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}(\log k)^2-\left(\frac{1}{n}\sum_{k=1}^{n}\log k\right)^2\right)=$

(a) 0 (b) 1 (c) 2 (d) 4

5. For positive a and b define,
$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$. For $a \in (0, 1)$,
$\int_0^\infty \frac{y^{a-1}}{1+y}\,dy=$

(a) $B(a, 1 - a)$ (b) $B(a, 1 + a)$
(c) $B\left(1-a, \frac{a}{2}\right)$ (d) $B\left(\frac{a}{2}, a\right)$

6. Suppose, f is continuous on $[a, b], f(a) = 0 = f(b)$ and $\int_a^b f^2(x)dx=1$. Then $\int_a^b xf(x)f'(x)dx=$

(a) 0 (b) 1/2 (c) $-1/2$ (d) 1

7. Suppose, f is continuous on $[0, c]$ and strictly increasing on $[0, c]$ with $f(0) = 0$, then for any $x \in [0, c]$, $\int_0^x f(t)dt+\int_0^{f(x)} f^{-1}(t)dt=$

(a) x (b) $f(x)$
(c) $xf(x)$ (d) $x + f(x)$

8. The function f is twice differentiable on $(0, \infty)$ with $f(5) = 3, f'(5) = 2$ and $\int_1^5 f(x)dx=5$, then $\int_1^5 (x-1)^2 f''(x)dx=$

(a) 15 (b) 18 (c) 20 (d) 0

9. For what value of a (> 1) $\int_a^{a^2}\frac{1}{x}\cdot\log\left(\frac{x-1}{32}\right)$ is minimum?

(a) 2 (b) 2.5 (c) 3 (d) 3.5

10. Let f satisfy the equation $x = f(x)\cdot e^{f(x)}$, then $\int_0^e f(x)dx=$

(a) 1 (b) $e - 1$ (c) $e + 1$ (d) 0

11. Call a function g as a lower-approximation for f on the interval $[a, b]$ if for all $x \in [a, b], f(x) \geq g(x)$. The maximum possible value of $\int_1^2 g(x)dx$, where

By : **ER Tapas Kr. Yogi,** (BHUBANESWAR) Mob : 09778158718

$g(x)$ is a linear lower-approximation for $f(x) = x^x$ on $[1, 2]$ is

(a) $\frac{3\sqrt{6}}{4}$ (b) $\frac{3\sqrt{6}}{3}$ (c) $\frac{3\sqrt{6}}{2}$ (d) $3\sqrt{6}$

12. $\int_0^1 (\sqrt{(x-1)^3+1} + x^{2/3} - (1-x)^{3/2} - \sqrt[3]{1-x^2})dx =$

(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/5

13. Suppose $f : R \to R$ be a continuous function which satisfies $f(-5) = 8, f(0) = 2$ and is even.

Let $g(x) = \begin{cases} f(x), & x \le 0 \\ 4 - f(x), & x > 0 \end{cases}$, then $\int_{-5}^{5} g(x)\,dx =$

(a) 0 (b) 10 (c) 20 (d) 40

14. Let D be the region in XY plane which is bounded by the parabola $y = 1 - x^2$ and the x-axis. For which +ve real no. c, does the parabola $y = cx^2$ divides D into three smaller regions of equal area?

(a) 1 (b) 2 (c) 4 (d) 8

15. $\int_0^1 \frac{dt}{\sqrt{1-t^4}} \div \int_0^1 \frac{dt}{\sqrt{1+t^4}} =$

(a) 1 (b) 2 (c) $\sqrt{2}$ (d) $2\sqrt{2}$

16. $f(x) = x, \quad x \in [-1, 1]$
$= x^3, \quad x \in (1, 2)$
$= x^2 + 4, \quad x \in [2, 3)$

(a) primitive of f for $x \in [-1, 1]$ is x^3.

(b) primitive of f for $x \in (1, 2)$ is $\frac{1}{4}x^4$.

(c) primitive of f for $x \in [2, 3)$ is $\frac{x^3}{3} + 4x - \frac{77}{12}$

(d) none of these

17. Let $f(x) = 1, x \in (0, 1]$
$= 0, x \in (1, 2]$

(a) primitive of $f(x)$ for $x \in (0, 1]$ is $x + c_1$.
(b) primitive of $f(x)$ for $x \in (1, 2]$ is $1 + c_1$.
(c) primitive of $f(x)$ for $x \in (1, 2]$ is c_2.
(d) no primitive of $f(x)$ is possible.

18. Given $\int_{-\infty}^{\infty} e^{k|x|}dx = 1$, then constant $k =$

(a) 1 (b) −1 (c) 2 (d) −2

19. $\lim_{n\to\infty} \binom{2n}{n}^{1/n} =$

(a) 1 (b) 2 (c) 1/2 (d) 4

20. For every $\theta \in (0, \pi]$, we have

(a) $\int_0^\theta \sqrt{1+\cos^2 t}\,dt > \sqrt{\theta^2 + \sin^2\theta}$

(b) $\int_0^\theta \sqrt{1+\cos^2 t}\,dt = \sqrt{\theta^2 + \sin^2\theta}$

(c) $\int_0^\theta \sqrt{1+\cos^2 t}\,dt < \sqrt{\theta^2 + \sin^2\theta}$

(d) $\int_0^\theta \sqrt{1+\cos^2 t}\,dt = \frac{1}{2}\sqrt{\theta^2 + \sin^2\theta}$

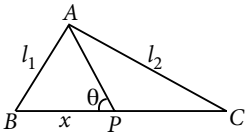
21. Let $n \ge 1$ be an integer. The real number $a \in (0, 1)$ that minimizes the integral $I(a) = \int_0^1 |x^n - a^n|\,dx$ is $a =$

(a) 1/2 (b) 1/3 (c) 1/4 (d) 2/3

22. $I_1 = \int_0^{\pi/2} \cos^8\theta d\theta$, $I_2 = \int_0^{\pi/2} \cos^6\theta d\theta$, then $\frac{I_1}{I_2} =$

(a) 7/8 (b) 8/7 (c) 8/9 (d) 9/8

23. In the diagram $l(AB) = l_1$, $l(AC) = l_2$, $l(BP) = x$ and $l(BC) = l$, then $\int_0^l (\cos\theta_{(x)})dx =$



(a) $l_2 + l_1$ (b) $l_2 - l_1$
(c) $2l_1 + l_2$ (d) $2l_1 - l_2$

24. $y(x)$ satisfies the differential equation $\frac{dy}{dx} = y\log y + ye^x$. If $y(0) = 1$, then $y(1) =$

(a) e^e (b) e^{-e} (c) $e^{1/e}$ (d) $\sqrt{e}$

25. $\int_0^1 ((e-1)\sqrt{\log(1+ex-x)} + e^{x^2})dx =$

(a) 1 (b) e (c) $1 + e$ (d) $e - 1$

26. Let $I=3\sqrt{2}\int_0^x \frac{\sqrt{1+\cos\theta}}{17-8\cos\theta}d\theta$. If $\theta \in (0, \pi)$ and $\tan I=\frac{2}{\sqrt{3}}$, then $x =$

(a) $\pi/3$ (b) $2\pi/3$ (c) $\pi/6$ (d) $\pi/2$

27. $\int_0^{\pi/2} (\sin^2(\sin x)+\cos^2(\cos x))dx =$

(a) $\pi/4$ (b) $\pi/2$ (c) π (d) 2π

28. The shortest possible length of an interval $[a, b]$for which $\int_a^b \frac{4}{1+x^2}dx=\pi$ is

(a) $\sqrt{2}$ (b) $2\sqrt{2}$

(c) $2\sqrt{2}+2$ (d) $2\sqrt{2}-2$

29. Suppose f and g are differentiable functions on R, satisfying $f(0) = g(0) = f'(0) = g'(0) = 1$ and

$f''(x)\ g(x) = 4f(x)\ g(x) - f'(x)g'(x)$

$g''(x)\ f(x) = 5f(x)\ g(x) - f'(x)\ g'(x)$.

If $f(1) = 1$, then $g(1) = ae^3 + be^{-3}$, where $a =$

(a) 5/6 (b) 2/3 (c) 6/5 (d) 3/2

30. Let $f : R \to R$ be a continuous non-constant periodic function of period T and let F denote an antiderivative of f. Let $g : R \to R$ be such that

$$F(x)=\left(\frac{1}{T}\int_0^T f(t)dt\right)x+g(x), \text{ then}$$

(a) $g(x)$ is periodic with period = T.

(b) $g(x)$ is periodic with period = $2T$.

(c) $g(x)$ is periodic with period = $T/2$.

(d) $g(x)$ is not periodic.

SOLUTIONS

1. (c) : In the inequality $\theta-\frac{\theta^3}{3!}<\sin\theta<\theta$, replace θ by $\frac{n}{n^2+k^2}$, $k=1$ to n.

and use $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}=\int_0^1\frac{dx}{1+x^2}=\frac{\pi}{4}$

2. (c) : Define

$$F(x)=x^2\cos\left(\frac{1}{x}\right)-2\int_0^x t\cos\left(\frac{1}{t}\right)dt,\ x\neq 0$$

$$= 0, \quad x = 0$$

Then $F'(x) = f(x)$ for $x \in R$ Thus F is an antiderivative of f in the case when $c = 0$.

3. (a) : For $0 \le \lambda_1 < \lambda_2 < 1$, $x \in (0, \pi/2)$

$-\lambda_1\cos^2 x > -\lambda_2\cos^2 x$

So, $\frac{1}{\sqrt{1-\lambda_1\cos^2 x}}<\frac{1}{\sqrt{1-\lambda_2\cos^2 x}}$

$f(\lambda_1) < f(\lambda_2)$, increasing.

4. (b) : Rewrite the term as

$$\frac{1}{n}\sum_{k=1}^{n-1}\left(\log\frac{k}{n}\right)^2-\left(\frac{1}{n}\sum_{k=1}^{n-1}\log\left(\frac{k}{n}\right)\right)^2$$

So, given limit $=\int_0^1(\log x)^2dx-\left(\int_0^1\log x dx\right)^2=1$

5. (a) : Substitute $x=\frac{y}{1+y}$ and $b = 1 - a$ in $B(a, b)$.

6. (c) : $\int_a^b x\,f(x)f'(x)dx=\frac{1}{2}\int_a^b x(f^2(x))'dx$

$$=\frac{1}{2}\left(xf^2(x)\Big|_a^b-\int_a^b f^2(x)dx\right)=-\frac{1}{2}$$

7. (c) : Put $u = f^{-1}(t)$ and integrate by parts,

$$\int_0^x f(t)dt+\int_0^{f(x)} f^{-1}(t)dt=\int_0^x f(t)dt+\int_0^x uf'(u)du$$

$$= x f(x)$$

8. (b) : Use integration by parts to get

$$\int_1^5(x-1)^2 f''(x)dx=(x-1)^2 f'(x)\Big|_1^5-2\int_1^5(x-1)f'(x)\,dx$$

$$=16f'(5)-2(x-1)f(x)\Big|_1^5+2\int_1^5 f(x)\,dx$$

$= 32 - 24 + 10 = 18$

9. (c) : Using Newton-Leibnitz formula,

$$I'(a)=\frac{(a+1)^2\log(a-1)-\log 32}{a}$$

So, for $a \in (1, 3)$, $I'(a) < 0$ and for $a > 3$. $I'(a) > 0$. So, minimum at $a = 3$.

10. (b) : Note that f is inverse function of $g(y) = ye^y$ and $f(e) = 1$

$$\int_0^e f(x)dx=e-\int_0^1 g(y)dy=e-\int_0^1 ye^y dy=e-1$$

11. (a) : Because g is linear, the area is actually a trapezium, with area $A = hm$, where m is the length of mid-line. For $g(x) \le f(x)$, $x \in [1, 2]$

$$m = g\left(\frac{3}{2}\right) \le f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^{3/2} = \frac{3\sqrt{6}}{4}$$

So, $\int_1^2 g(x)dx \le hm = \frac{3\sqrt{6}}{4}$

12. (d) : Notice that $\int_0^1 \sqrt{(x-1)^3+1} - \sqrt[3]{1-x^2}\,dx = 0$

inverse functions and shifted.

So, given integral $= \int_0^1 x^{2/3} - (1-x)^{3/2}dx = \frac{1}{5}$

13. (c) : Define $h(x) = g(x) - 2$ and notice that $h(x)$ is odd.

So, $\int_{-5}^{5} g(x) = \int_{-5}^{5} h(x) + 2 = \int_{-5}^{5} h(x) + \int_{-5}^{5} 2dx = 20$

14. (d) : Area of $D = 2\int_0^1 (1-x^2)dx = \frac{4}{3}$.

The two curves intersect at $x^2 = \frac{1}{1+c}$. Hence

$2\int_0^{\frac{1}{\sqrt{1+c}}} ((1-x^2) - cx^2)dx = \frac{1}{3} \times \frac{4}{3}$ gives $c = 8$

15. (c) : For $\int_0^1 \frac{dt}{\sqrt{1-t^4}}$, put $t^2 = \sin\theta$ to simplify to

$\frac{1}{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$. For $\int_0^1 \frac{dt}{\sqrt{1+t^4}}$, put $t^2 = \tan(\beta/2)$ to simplify to $\frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{d\beta}{\sqrt{\sin\beta}}$.

16. (c) : Any primitive of f is of the following form,

$$F(x) = \frac{x^2}{2} + c_1, \quad x \in [-1, 1]$$
$$= \frac{x^4}{4} + c_2, \quad x \in (1, 2)$$
$$= \frac{x^3}{3} + 4x + c_3, \quad x \in [2, 3)$$

For F to be continuous, $\frac{1}{2} + c_1 = \frac{1}{4} + c_2$ and

$\frac{16}{4} + \frac{1}{4} + c_1 = \frac{8}{3} + 8 + c_3$

17. (d) : Any primitive of f is of the form

$F(x) = x + c_1, \quad x \in [0, 1]$
$= c_2, \quad x \in (1, 2]$

So, F continuity gives $1 + c_1 = c_2$

So, $F(x) = x + c_1$, $x \in [0, 1]$ $= 1 + c_2$, $x \in (1, 2]$

But this function is not differentiable.

18. (d) : $\int_{-\infty}^{\infty} e^{k|x|}dx = 2\int_0^{\infty} e^{kx}dx = \lim_{x\to\infty}\left(\frac{2}{k}e^{kx} - \frac{2}{k}\right) = 1$ (given)

So, $k < 0$ and $0 - \frac{2}{k} = 1 \Rightarrow k = -2$

19. (d) : Notice that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+2)!}{((n+1)!)^2} \times \frac{(n!)^2}{(2n)!}$$

where $a_n = {}^{2n}C_n$

$$= \lim_{n\to\infty}\frac{(2n+1)(2n+2)}{(n+1)^2} = 4$$

and $\lim_{n\to\infty} a_n^{1/n} = 4$

20. (a) : Notice that $\int_0^{\theta} \sqrt{1+\cos^2 t}\,dt$ is the arc length of the curve $y = \sin x$ from $(0, 0)$ to $(\theta, \sin\theta)$ and $\sqrt{\theta^2 + \sin^2\theta}$ is the normal distance between these same points.

21. (a) : Since $f(x) = x^n$ is an increasing function on $[0, 1]$.

$$I(a) = \int_0^1 |x^n - a^n|\,dx = \int_0^a (a^n - x^n)dx + \int_a^1 (x^n - a^n)dx$$
$$= \frac{2n}{(n+1)}a^{n+1} - a^n + 1$$

Now, $\frac{d(I(a))}{da} = 0$ gives $a = \frac{1}{2}$

22. (a) : $I = \int_0^{\pi/2} \cos^6\theta(1 - \sin^2\theta)d\theta$

$= J - \int_0^{\pi/2} \cos^6\theta\sin^2\theta\,d\theta$ and $\int_0^{\pi/2} \cos^6\theta\sin^2\theta\,d\theta$

can be evaluated by putting $\cos^7\theta = t$.

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MOCK TEST PAPER 2015

JEE Main

Exam on 4th April 2015

* **ALOK KUMAR, B.Tech, IIT Kanpur**

1. If $a_1, a_2, a_3, \ldots, a_{4001}$ are terms of an A.P. such that $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots.+\frac{1}{a_{4000}a_{4001}}=10$ and $a_2 + a_{4000} = 50$, then $|a_1 - a_{4001}|$ is equal to

(a) 20 (b) 30 (c) 40 (d) 10

2. From a point P, perpendicular tangents PQ and PR are drawn to ellipse $x^2 + 4y^2 = 4$. Locus of circumcentre of triangle PQR is

(a) $x^2+y^2=\frac{16}{5}(x^2+4y^2)^2$

(b) $x^2+y^2=\frac{5}{16}(x^2+4y^2)^2$

(c) $x^2+4y^2=\frac{16}{5}(x^2+y^2)^2$

(d) $x^2+4y^2=\frac{5}{16}(x^2+y^2)^2$

3. Let $F(x)=\sin x\int_0^x \cos t\,dt+2\int_0^x t\,dt+\cos^2 x-x^2$. Then area bounded by $xF(x)$ and ordinate $x = 0$ and $x = 5$ with x-axis is

(a) 16 (b) $\frac{25}{2}$ (c) $\frac{35}{2}$ (d) 25

4. Let a function $f(x)$ be such that $f''(x) = f'(x) + e^x$ and $f(0) = 0, f'(0) = 1$, then $\ln\left(\frac{(f(2))^2}{4}\right)$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4

5. If the line $y = x + 2$ does not intersect any member of family of parabolas $y^2 = ax$, $(a \in R^+)$ at two distinct points, then maximum value of latus rectum of parabola is

(a) 4 (b) 8 (c) 16 (d) 32

6. Equation of circle inscribed in $|x - a| + |y - b| = 1$ is

(a) $(x + a)^2 + (y + b)^2 = 2$

(b) $(x - a)^2 + (y - b)^2 = 1/2$

(c) $(x - a)^2 + (y - b)^2 = 1/\sqrt{2}$

(d) $(x - a)^2 + (y - b)^2 = 1$

7. The number of terms in $(a_1 + a_2 + a_3 + a_4)^3$ is

(a) 64 (b) 81 (c) 30 (d) 20

8. The number of functions f from the set $A = \{0, 1, 2\}$ into the set $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \le f(j)$ for $i < j$ and $i, j \in A$ is

(a) ${}^8C_3$ (b) ${}^8C_3 + 2({}^8C_2)$

(c) ${}^{10}C_3$ (d) ${}^{10}C_4$

9. Eight straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. The number of parts into which these lines divide the plane is

(a) 29 (b) 32 (c) 36 (d) 37

10. If $\sum_{i=1}^{n}\vec{a_i}=\vec{0}$ where $|\vec{a_i}|=1\,\forall i$, then the value of $\sum_{1\le i<\,j\le n}\sum \vec{a_i}\cdot\vec{a_j}$ is

(a) $-\frac{n}{2}$ (b) $-n$ (c) $\frac{n}{2}$ (d) n

11. If the system of linear equations $x + y + z = 6$, $x + 2y + 3z = 14$ and $2x + 5y + \lambda z = \mu$, $(\lambda, \mu \in R)$ has no solution, then

(a) $\lambda \ne 8$ (b) $\lambda = 8, \mu \ne 36$

(c) $\lambda = 8, \mu = 36$ (d) None of these

12. A man starts from the point $P(3, -3)$ and reaches the point $Q(0, 2)$ after touching the line $2x + y = 7$ at R. The least value of $PR + RQ$ is

(a) $5\sqrt{2}$ (b) $3\sqrt{2}$ (c) $7\sqrt{2}$ (d) $2\sqrt{2}$

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13. A hyperbola passing through origin has $3x - 4y - 1 = 0$ and $4x - 3y - 6 = 0$ as its asymptotes. Then the equation of its transverse axis is

(a) $x - y - 5 = 0$ (b) $x + y + 1 = 0$

(c) $x + y - 5 = 0$ (d) $x - y - 1 = 0$

14. Let $f\left(\frac{x+y}{2}\right)=\frac{1}{2}(f(x)+f(y))$ for real x and y. If $f'(x)$ exists and equals to -1 and $f(0) = 1$, then the value of $f(2)$ is

(a) 1 (b) -1 (c) $\frac{1}{2}$ (d) 2

15. $\lim\limits_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}$ equals

(a) e (b) e^{-1} (c) e^{-2} (d) e^2

16. If $|\tan A| < 1$ and $|A|$ is acute, then $\frac{\sqrt{1+\sin 2A}+\sqrt{1-\sin 2A}}{\sqrt{1+\sin 2A}-\sqrt{1-\sin 2A}}$ is equal to

(a) $\tan A$ (b) $-\tan A$ (c) $\cot A$ (d) $-\cot A$

17. In ΔABC, orthocentre is (6,10), circumcentre is (2, 3) and equation of side $\overleftrightarrow{BC}$ is $2x + y = 17$. Then the radius of the circumcircle of ΔABC is

(a) 4 (b) 5 (c) 7 (d) 3

18. The inclination to the major axis of the diameter of an ellipse, the square of whose length is the harmonic mean between the squares of the major and minor axes is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{2}$

19. The value of $\lim\limits_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$ is

(a) $\frac{e}{2}$ (b) $-\frac{e}{2}$ (c) $\frac{3e}{2}$ (d) $-\frac{2e}{3}$

20. If $I=\int\frac{dx}{\sin\left(x-\frac{\pi}{3}\right)\cos x}$, then I equals

(a) $2\log\left|\sin x+\sin\left(x-\frac{\pi}{3}\right)\right|+C$

(b) $2\log\left|\sin\left(x-\frac{\pi}{3}\right)\sec x\right|+C$

(c) $2\log\left|\sin x-\sin\left(x-\frac{\pi}{3}\right)\right|+C$

(d) None of these

21. Let $A_1, A_2, A_3, \ldots, A_{40}$ are 40 sets each with 7 elements and $B_1, B_2, \ldots, B_n$ are n sets each with 7 elements. If $\bigcup\limits_{i=1}^{40}A_i=\bigcup\limits_{j=1}^{n}B_j=S$ and each element of S belongs to exactly ten of A_i's and exactly 9 of B_j's, then n equals

(a) 42 (b) 35 (c) 28 (d) 36

22. If x, y, z are distinct positive numbers, then $x^{\ln y-\ln z}+y^{\ln z-\ln x}+z^{\ln x-\ln y}\in$

(a) $(0,\infty)$ (b) $(1,\infty)$ (c) $(3,\infty)$ (d) $(1, 3)$

23. Let PQ be a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which subtends an angle of $\pi/2$ radians at the centre. If L is the foot of perpendicular from (0, 0) to PQ, then

(a) locus of L is an ellipse

(b) locus of L is circle concentric with given ellipse

(c) locus of L is a hyperbola concentric with given ellipse

(d) a square concentric with given ellipse

24. A natural number is selected from 1 to 100 so that the probability, if it satisfies the number $\frac{x^2-60x+800}{x-30}<0$ is

(a) 7/25 (b) 4/25 (c) 2/25 (d) 8/25

25. The digit at unit place in the number $17^{1995} + 11^{1995} - 7^{1995}$ is

(a) 0 (b) 1 (c) 2 (d) 3

26. The range of values of the term independent of x in the expansion of $\left[x\sin^{-1}\alpha+\frac{\cos^{-1}\alpha}{x}\right]^{10}$, $\alpha\in[-1, 1]$ is

(a) $\left[\frac{-^{10}C_5\,\pi^{10}}{2^5},\frac{^{10}C_5\,\pi^{20}}{2^{20}}\right]$

(b) $\left[\frac{^{10}C_5\,\pi^{2}}{2^{20}},\frac{-^{10}C_5\,\pi^{2}}{2^{20}}\right]$

(c) [1, 2] (d) (1, 2)

27. Area of trapezium whose vertices lie on the parabola $y^2 = 4x$ and its diagonals pass through (1, 0) and having length 25/4 units each , is

(a) $\frac{75}{4}$ sq. units (b) $\frac{625}{16}$ sq. units

(c) $\frac{25}{4}$ sq. units (d) $\frac{25}{8}$ sq. units

28. If the sum of the slopes of the normal from point 'P' to the hyperbola $xy = c^2$ is equal to $\lambda(\lambda \in R^+)$, then locus of point 'P' is

(a) $x^2 = \lambda c^2$ (b) $y^2 = \lambda c^2$

(c) $xy = \lambda c^2$ (d) none of these

29. The differential equation of all straight lines which are at a constant distance 'p' from the origin is

(a) $y - x\frac{dy}{dx} = p^2\left[1 + \frac{d^2y}{dx^2}\right]$

(b) $y - x\frac{dy}{dx} = p\left[1 + \left(\frac{dy}{dx}\right)^2\right]$

(c) $\left(y - x\frac{dy}{dx}\right)^2 = p^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$

(d) $\left(y - x\frac{dy}{dx}\right)^2 = p\left[1 + \left(\frac{dy}{dx}\right)^2\right]$

ANSWER KEYS

1. (b)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (b)
6. (b)	**7.** (d)	**8.** (c)	**9.** (d)	**10.** (a)
11. (b)	**12.** (a)	**13.** (c)	**14.** (b)	**15.** (b)
16. (c)	**17.** (b)	**18.** (a)	**19.** (b)	**20.** (b)
21. (d)	**22.** (c)	**23.** (b)	**24.** (a)	**25.** (b)
26. (b)	**27.** (a)	**28.** (a)	**29.** (c)	

Contd. from Page No. 13

PRACTICE PAPER 2015
JEE Main
Integral Calculus

23. (b) : Let $l(AP) = y$, using cosine formula in both the small triangles,

$l_2^2 = (l - x)^2 + y^2 + 2(l - x)y\cos\theta$ and

$l_1^2 = x^2 + y^2 - 2xy\cos\theta$

Eliminating $\cos\theta$ from these two equations

$$y^2 = l_1^2 - x^2 + \left(\frac{l_2^2 - l_1^2 + 2lx - l^2}{l}\right)x$$

Doing $\frac{dy}{dx}$ amounts to $\frac{dy}{dx} = \cos\theta$.

24. (a) : Put $y = e^u$ to convert the given differential equation into a linear differential equation.

25. (b) : Transform the integral

$\int_0^1 (e-1)\sqrt{\log(1+ex-x)}\,dx$ to $\int_1^e \sqrt{\log y}\,dy$ which is inverse of $y = e^{x^2}$.

26. (b) : First put $t = 2y$ and then $z = \sin y$.

27. (b) : $I = \int_0^{\pi/2} (\sin^2(\sin x) + \cos^2(\cos x))dx$ ...(i)

$I = \int_0^{\pi/2} (\sin^2(\cos x) + \cos^2(\sin x))dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} (1+1)dx$. So, $I = \pi/2$

28. (d) : $\tan^{-1} b - \tan^{-1} a = \frac{\pi}{4}$

Simplifying, $b = \frac{a+1}{1-a}$

So, length of interval $l = b - a = \frac{1+a^2}{1-a}$

Using, $\frac{dl}{da} = 0$ gives $a = 1 - \sqrt{2}$ suitably

29. (a) : Adding the two equations, give

$f''(x)\,g(x) + 2f'(x)\,g'(x) + g''(x)\,f(x) = 9\,f(x)\,g(x)$

Define $h(x) = f(x)\,g(x)$, then the above equation becomes $h''(x) = 9h(x)$

30. (a) : Notice that $F(x + T) - F(x)$

$= \int_0^T f(t)dt = h(x+T) - h(x)$,

where $h(x) = \left(\frac{1}{T}\int_0^T f(t)dt\right)x$

So, $g(x) = F(x) - h(x) = F(x + T) - h(x + T)$ is periodic with period = T.

ENRICH YOUR CONCEPTS ON XI

Statistics & Probability

Based on NCERT Pattern

Series-8

STATISTICS

DEFINITION

A type of mathematical analysis involving the use of quantified representation, models and summaries for a given set of empirical data or real world observations. Statistical analysis involves the process of collecting and analyzing data and then summarizing the data into a numerical form.

MEASURES OF CENTRAL TENDENCY

It is a measure that tells us where the middle of a bunch of data lies. The three most common measures of central tendency are mean, median and mode.

MEASURES OF DISPERSION

It is possible to have two very different data sets with the same means and medians. For that reason, measures of the middle are useful but limited. So, dispersion or variability is a very important attribute of a data set. It measures the degree of scatteredness of the observations in a distribution around the central value.

The various measures of dispersion are

(a) Range (b) Quartile Deviation

(c) Mean Deviation (d) Standard Deviation

RANGE

It is the difference between two extreme values *i.e.*, Range = Maximum value – Minimum value.

MEAN DEVIATION

The mean deviation from a central value 'a' is given by,

M.D. (a)

$$= \frac{\text{Sum of absolute values of deviations from } a}{\text{Number of observations}}$$

1. Mean deviation for ungrouped data

Let $x_1, x_2, \ldots, x_n$ be n observations, then

About mean, $\bar{x}$	$\frac{1}{n}\sum_{i=1}^{n} \lvert x_i - \bar{x} \rvert$
About median, M	$\frac{1}{n}\sum_{i=1}^{n} \lvert x_i - M \rvert$

2. Mean deviation for grouped data

(i) For discrete frequency distribution : Let $x_1, x_2, \ldots, x_n$ be a set of n observations occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively, then

About mean, $\bar{x}$	$\frac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - \bar{x} \rvert$
About Median, M	$\frac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - M \rvert$

where, $N = \sum_{i=1}^{n} f_i$ = Sum of frequencies.

(ii) For continuous frequency distribution: Here x_i are the mid-points of the classes.

About mean, $\bar{x}$	$\frac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - \bar{x} \rvert$
About Median, M	$\frac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - M \rvert$

Shortcut method for calculating mean deviation

- About mean, $\bar{x} = A + \frac{\left(\sum_{i=1}^{n} f_i d_i\right)}{N} \times h$, where '$A$' is the assumed mean and $d_i = \frac{x_i - A}{h}$.

$$\therefore \quad \text{M.D.}(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - \bar{x}|$$

- About median, $M = l + \dfrac{\frac{N}{2} - C}{f} \times h$

  where median class is the class interval whose cumulative frequency is just greater than or equal to $\frac{N}{2}$, N is sum of frequencies, l, f, h and C are respectively the lower limit, the frequency, the width of the median class and the cumulative frequency of the class just preceding the median class.

$$\text{M.D.}(M) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - M|$$

VARIANCE

The mean of the squares of the deviations from mean is called the variance and is denoted by σ^2.

For ungrouped data	$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$
For grouped data	$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2$

STANDARD DEVIATION

The proper measure of dispersion about the mean of a set of observations is expressed as positive square root of the variance, called standard deviation and is denoted by σ.

For ungrouped data	$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$
For a discrete frequency distribution	$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$
For continuous frequency distribution	$\sigma = \frac{1}{N}\sqrt{N\sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2}$

Shortcut Method for finding standard deviation

$$\sigma = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2},$$

where $y_i = \dfrac{x_i - A}{h}$, A = assumed mean, h = width of class intervals.

COEFFICIENT OF VARIATION

- The measure of variability which is independent of units, is called coefficient of variation and usually denoted by C.V.

  $$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100, \bar{x} \neq 0,$$

  where σ = standard deviation, $\bar{x}$ = mean.
- Comparing the variability or dispersion of two series, we calculate the coefficient of variance for each series. The series having greater C.V. is said to be more variable than the other.

  The series having lesser C.V. is said to be more consistent than the other.
- For two series with equal means, the series with greater standard deviation (or variance) is called more variable or dispersed than the other. Also, the series with lesser value of standard deviation (or variance) is said to be more consistent than the other.
- **Variance of two groups of observations taken together**

  Let us consider two groups of variates containing n_1 and n_2 items with respective means $\bar{x}_1$ and $\bar{x}_2$. Let the standard deviations for the two groups be σ_1 and σ_2 respectively. Let σ be the standard deviation of $n_1 + n_2$ items taken together and $\bar{x}$ be the mean of two groups taken together, then

  $$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

  $$\therefore \quad \sigma^2 = \frac{1}{n_1 + n_2}\left[n_1\sigma_1^2 + n_2\sigma_2^2 + \frac{n_1 n_2}{(n_1 + n_2)}(\bar{x}_1 - \bar{x}_2)^2\right]$$

PROBABILITY

DEFINITION

A measure of uncertainty or estimation of occurrence of an event is called probability. Probabilities are given a value between 0 (0% chance or event will not happen) and 1 (100% chance or event will happen).

TERMS USED IN PROBABILITY

- **Random Experiment :** An experiment is said to be random experiment if it has more than one possible outcomes and it is not possible to predict the outcome in advance.
- **Outcome :** A possible result of a random experiment is called its outcome.
- **Sample Space :** The set of all possible outcomes of an experiment is called sample space.
- **Sample Points :** Elements of sample space are called sample points.
- **Event :** A subset of a sample space is called an event.
- **Occurrence of an Event :** An event associated to a random experiment is said to occur if any one of the elementary events associated to it is an outcome.

TYPES OF EVENT

Impossible Event	The empty set 'ϕ'.
Sure Events	The whole sample space.
Simple or Elementary Event	If an event E has only one sample point of a sample space.
Compound Event	If an event E has more than one sample point.

ALGEBRA OF EVENTS

Complementary Event	For every event A, complement of A is the event consisting of all sample space outcomes that do not correspond to the occurrence of A. It is also called the event 'not A' or $S - A$.
Event 'A or B'	If A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both.' This event '$A \cup B$' is called 'A or B'. $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$
Event 'A and B'	If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'. $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$
Event 'A but not B'	$A - B$ is the set of all those elements which are in A but not in B. Therefore, the set $A - B$ may denote the event 'A but not B'. $A - B = A \cap B'$
Mutually Exclusive Events	Two events A and B of a sample space S are said to be mutually exclusive if the occurrence of any one of them excludes the occurrence of the other. Hence, $A \cap B = \phi$ *i.e.* A and B are disjoint sets.
Exhaustive Events	Let $E_1, E_2, E_3, \ldots, E_n$ be n events of a sample space S. If $E_1 \cup E_2 \cup \ldots \ldots \cup E_n = \bigcup_{i=1}^{n} E_i = S$, then $E_1, E_2, \ldots, E_n$ are called exhaustive events.
Mutually exclusive and exhaustive events	If $E_i \cap E_j = \phi$ for $i \neq j$ *i.e.*, events E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, \ldots, E_n$ are called mutually exclusive and exhaustive events.

VERY SHORT ANSWER TYPE (1 MARK)

1. Find the range of the data, 35, 50, 48, 62, 27, 39, 43, 72, 56, 68.

2. If the S.D. of the items x_1, x_2, x_3, ..., x_n is σ, what is the S.D. of the data $x_1 + a$, $x_2 + a$, $x_3 + a$, ..., $x_n + a$?

3. A fair die is rolled once. Find the probability that a number less than 7 shows up.

4. A book contains 100 pages. A page is chosen at random. What is the probability that the sum of the digits on the page is equal to 9?

5. Three unbiased coins are tossed once. What is the probability of getting atleast 1 head?

SHORT ANSWER TYPE (4 MARKS)

6. Two cards are drawn at random from a pack of 52 cards. What is the probability that both the cards are aces?

7. The mean and variance of the heights and weights of the students of a class are given below.

	Height	Weight
Mean	160 cm	50.4 kg
Variance	116.64 cm^2	17.64 kg^2

Show that the weights are more variable than heights.

8. Find the mean deviation about mean for the following data.

x_i	3	5	7	9	11	13
f_i	6	8	15	25	8	4

9. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that it being a spade or a king.

10. Find the mean deviation about the median for the data given below.
45, 36, 50, 60, 53, 46, 51, 48, 72, 42

LONG ANSWER TYPE (6 MARKS)

11. Find the mean, variance and standard deviation for the following data using short cut method.

x_i	60	61	62	63	64	65	66	67	68
f_i	2	1	12	29	25	12	10	4	5

12. The following is the record of goals scored by team *A* in football session.

Number of goals scored	0	1	2	3	4
Number of matches	1	9	7	5	3

For the team *B*, mean number of goals scored per match was 2 with a standard deviation 1.25 goals. Find which team may be considered more consistent.

13. Four persons are to be chosen at random from a group of 3 men, 2 women and 4 children. Find the probability of selecting
(i) 1 man, 1 woman and 2 children.
(ii) exactly 2 children.
(iii) exactly 2 women.

14. Five marbles are drawn from a bag which contains 7 blue marbles and 4 black marbles. What is the probability that
(i) all will be blue?
(ii) 3 will be blue and 2 black?

15. Calculate the mean and standard deviation for the following table given the age distribution of a group of people.

Age	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of persons	3	51	122	141	130	51	2

SOLUTIONS

1. Range = 72 – 27 = 45.

2. Remains unchanged.

3. Sample space $S = \{1, 2, 3, 4, 5, 6,\}$.
Let E be the event that 'the number less than 7 shows up', then $E = \{1, 2, 3, 4, 5, 6\} = S$
$\therefore \quad P(E) = P(S) = 1$

4. Sample space $S = \{1, 2, ..., 100\}$
$\therefore \quad n(S) = 100$

Let E be the event that "the sum of the digits on the page is equal to 9".

$\Rightarrow$ $E = \{9, 18, 27, 36, 45, 54, 63, 72, 81, 90\}$

$\therefore$ $n(E) = 10$

So, required probability $= \dfrac{n(E)}{n(S)} = \dfrac{10}{100} = \dfrac{1}{10}$

5. Sample space $S = \{HTT, THT, TTH, TTT, HHT, HTH, THH, HHH\}$

$\therefore$ $n(S) = 8$

Let E = event of getting atleast 1 head. Then,

$E = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}$.

$\therefore$ $n(E) = 7$.

$\therefore$ $P(\text{getting atleast 1 head}) = \dfrac{n(E)}{n(S)} = \dfrac{7}{8}$

6. Let S be the sample space. Then,

$n(S)$ = number of ways of selecting 2 cards out of 52

$= {}^{52}C_2 = \dfrac{(52 \times 51)}{(2 \times 1)} = 1326.$

Let E = event that both the cards are aces. Then,

$n(E)$ = number of ways of drawing 2 aces out of 4

$= {}^{4}C_2 = \dfrac{(4 \times 3)}{(2 \times 1)} = 6.$

$P(\text{getting 2 aces}) = P(E) = \dfrac{n(E)}{n(S)} = \dfrac{6}{1326} = \dfrac{1}{221}$

7. For height, we have

Variance, $\sigma^2 = 116.64 \text{ cm}^2$

$\Rightarrow$ S.D., $\sigma = \sqrt{116.64} \text{ cm} = 10.8 \text{ cm}$

Also, mean = 160 cm

$\therefore$ $\text{C.V.}_1 = \left(\dfrac{\text{S.D.}}{\text{mean}} \times 100\right) = \left(\dfrac{10.8}{160} \times 100\right) = 6.75$

For weight, we have

Variance, $\sigma^2 = 17.64 \text{ kg}^2$

$\Rightarrow$ S.D., $\sigma = \sqrt{17.64} \text{ kg} = 4.2 \text{ kg}$

And, mean = 50.4 kg

$\therefore$ $\text{C.V.}_2 = \left(\dfrac{\text{S.D.}}{\text{mean}} \times 100\right) = \left(\dfrac{4.2}{50.4} \times 100\right) = 8.33$

Clearly, (C.V. of weights) > (C.V. of heights).

Hence, weights are more variable than heights.

8. We prepare the table given below

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
3	6	18	5	30
5	8	40	3	24
7	15	105	1	15
9	25	225	1	25
11	8	88	3	24
13	4	52	5	20
Total	66	528	-	138

$\therefore$ $\bar{x} = \dfrac{\sum_{i=1}^{6} f_i x_i}{N} = \dfrac{528}{66} = 8$

$\therefore$ $\text{M.D.}(\bar{x}) = \dfrac{\sum_{i=1}^{6} f_i \lvert x_i - \bar{x} \rvert}{N} = \dfrac{138}{66} = 2.09$

9. Let S be the sample space. Then, $n(S) = 52$

Let E_1 = event of getting a spade,

and E_2 = event of getting a king.

Then, $E_1 \cap E_2$ = event of getting a king of spade.

Clearly, $n(E_1) = 13$, $n(E_2) = 4$ and $n(E_1 \cap E_2) = 1$

$\therefore$ $P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{13}{52} = \dfrac{1}{4}$,

$P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{4}{52} = \dfrac{1}{13}$

and $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{1}{52}$

$\therefore$ $P(\text{getting a spade or a king}) = P(E_1 \text{ or } E_2)$

$= P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \left(\dfrac{1}{4} + \dfrac{1}{13} - \dfrac{1}{52}\right) = \dfrac{16}{52} = \dfrac{4}{13}$

10. Arranging the given data in ascending order, we get

36, 42, 45, 46, 48, 50, 51, 53, 60, 72

Here $n = 10$, which is even.

$\therefore$ Median $(M) = \frac{1}{2}\left\{\left(\frac{n}{2}\right)^{\text{th}} \text{ observation} + \left(\frac{n}{2}+1\right)^{\text{th}} \text{ observation}\right\}$

$\Rightarrow$ $M = \frac{1}{2}(5^{\text{th}} \text{ observation} + 6^{\text{th}} \text{ observation})$

$\Rightarrow$ $M = \frac{1}{2}(48+50) = \frac{98}{2} = 49$

$\therefore$ The values of $(x_i - M)$ are

$-13, -7, -4, -3, -1, 1, 2, 4, 11, 23$

$\therefore$ $\sum_{i=1}^{10} |x_i - M| = (13 + 7 + 4 + 3 + 1 + 1 + 2 + 4 + 11 + 23) = 69$

$\therefore$ M.D. $(M) = \frac{\sum_{i=1}^{10} |x_i - M|}{n} = \frac{69}{10} = 6.9$

11. Let the assumed mean be $A = 64$.

Then, we prepare the table given below.

x_i	f_i	$d_i = (x_i - 64)$	d_i^2	$f_i d_i$	$f_i d_i^2$
60	2	– 4	16	– 8	32
61	1	–3	9	– 3	9
62	12	– 2	4	–24	48
63	29	– 1	1	– 29	29
64 = A	25	0	0	0	0
65	12	1	1	12	12
66	10	2	4	20	40
67	4	3	9	12	36
68	5	4	16	20	80
Total	100			0	286

$\therefore$ $N = \sum f_i = 100, \sum f_i d_i = 0$ and $\sum f_i d_i^2 = 286$

$\therefore$ Mean $= A + \frac{\sum f_i d_i}{N} = \left(64 + \frac{0}{100}\right) = 64$

Variance, $\sigma^2 = \left(\frac{\sum f_i d_i^2}{N} - \left(\frac{\sum f_i d_i}{N}\right)^2\right)$

$= \frac{286}{100} - \left(\frac{0}{100}\right)^2 = 2.86$

$\therefore$ Standard deviation, $\sigma = \sqrt{2.86} = 1.69$

12. In order to determine the consistency of teams, we will have to find the coefficients of variations of two teams.

Computation of mean and standard deviation of goals scored by team A.

x_i	f_i	$f_i x_i$	$f_i x_i^2$
0	1	0	0
1	9	9	9
2	7	14	28
3	5	15	45
4	3	12	48
Total	25	50	130

$\therefore$ $N = \sum f_i = 25, \sum f_i x_i = 50$ and $\sum f_i x_i^2 = 130$

$\therefore$ $\bar{x}_A = \frac{1}{N}\left\{\sum f_i x_i\right\} = \frac{50}{25} = 2$

and $\sigma_A^2 = \left(\frac{1}{N}\sum f_i x_i^2 - \left\{\frac{1}{N}\sum f_i x_i\right\}^2\right)$

$= \frac{130}{25} - \left(\frac{50}{25}\right)^2 = 5.2 - 4 = 1.2$

$\Rightarrow$ $\sigma_A = \sqrt{1.2} = 1.095$

It is given that $\bar{x}_B = 2$ and $\sigma_B = 1.25$

Now,

Coefficient of variation of goals scored by team A

$= \frac{\sigma_A}{\bar{x}_A} \times 100 = \frac{1.095}{2} \times 100 = 54.75$

Coefficient of variation of goals scored by team B

$= \frac{\sigma_B}{\bar{x}_B} \times 100 = \frac{1.25}{2} \times 100 = 62.50$

We observe that, the coefficient of variation of goals scored by team A is lesser than that of team B. Hence, team A is more consistent.

13. There are 9 persons *i.e.*, 3 men, 2 women and 4 children. Out of these 9 persons, 4 persons can be selected in ${}^9C_4 = 126$ ways.

$\therefore$ Total number of elementary events = 126

(i) 1 man, 1 woman and 2 children can be selected in ${}^3C_1 \times {}^2C_1 \times {}^4C_2 = 36$ ways.

$\therefore$ Favourable number of elementary events $= 36$

So, required probability $= \frac{36}{126} = \frac{2}{7}$

(ii) Exactly 2 children means 2 children and 2 persons from 3 men and 2 women. This can be done in ${}^4C_2 \times {}^5C_2 = 60$ ways.

$\therefore$ Favourable number of elementary events $= 60$

So, required probability $= \frac{60}{126} = \frac{10}{21}$

(iii) We have to select 4 persons of which 2 are women and the remaining 2 are chosen from 3 men and 4 children. This can be done in ${}^2C_2 \times {}^7C_2 = 21$ ways.

$\therefore$ Favourable number of elementary events $= 21$

So, required probability $= \frac{21}{126} = \frac{1}{6}$

14. There are $7 + 4 = 11$ marbles in the bag out of which 5 marbles can be drawn in ${}^{11}C_5$ ways.

$\therefore$ Total number of elementary events $= {}^{11}C_5$

(i) There are 7 blue marbles out of which 5 blue marbles can be drawn in ${}^7C_5$ ways.

$\therefore$ Favourable number of elementary events $= {}^7C_5$

Hence, required probability $= \frac{{}^7C_5}{{}^{11}C_5}$

$$= \frac{7!}{2!5!} \times \frac{5!6!}{11!} = \frac{1}{22}$$

(ii) 3 blue out of 7 blue marbles and 2 black out of 4 black marbles can be drawn in ${}^7C_3 \times {}^4C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^7C_3 \times {}^4C_2$

Hence, required probability

$$= \frac{{}^7C_3 \times {}^4C_2}{{}^{11}C_5} = \frac{7!}{3!4!} \times \frac{4!}{2!2!} \times \frac{5!6!}{11!} = \frac{5}{11}$$

15. Here $A = 55$, $h = 10$.

Age	x_i	f_i	$u_i = \frac{x_i - 55}{10}$	$f_i u_i$	u_i^2	$f_i u_i^2$
20-30	25	3	–3	–9	9	27
30-40	35	51	–2	–102	4	204
40-50	45	122	–1	–122	1	122
50-60	55	141	0	0	0	0
60-70	65	130	1	130	1	130
70-80	75	51	2	102	4	204
80-90	85	2	3	6	9	18
Total		500		5		705

$$\bar{x} = A + h\left(\frac{1}{N}\Sigma f_i u_i\right) = 55 + 10\left(\frac{5}{500}\right) = 55.1$$

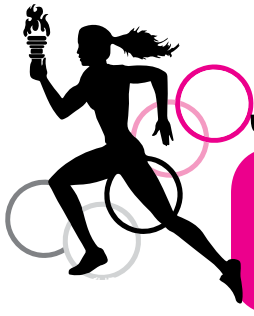
$$\sigma^2 = h^2\left[\frac{1}{N}\Sigma f_i u_i^2 - \left(\frac{1}{N}\Sigma f_i u_i\right)^2\right]$$

$$= 100\left[\frac{705}{500} - \left(\frac{5}{500}\right)^2\right] = \frac{14099}{100}$$

$$\Rightarrow \quad \sigma = \frac{\sqrt{14099}}{10} = \frac{118.739}{10} = 11.8739$$

Important Exam Dates 2015

- JEE (Main-Offline) : 4th April
- JEE (Main-Online) : 10th - 11th April
- VITEEE : 8th - 19th April
- WB JEE : 18th - 19th April
- Kerala PET : 20th - 21st April
- COMED K : 10th May
- UPSEAT : 16th May
- BVP (E) : 17th May
- JEE Advanced : 24th May



JUNIOR MATHEMATICAL OLYMPIAD

1. Prove that for no integer n, $n^6 + 3n^5 - 5n^4 - 15n^3 + 4n^2 + 12n + 3$ is a perfect square.

2. Two dice are thrown once simultaneously. Let E be the event "Sum of numbers appearing on the dice." What are the members of E? Can you load these dice (not necessarily in the same way) such that all members of E are equally likely? Give justification.

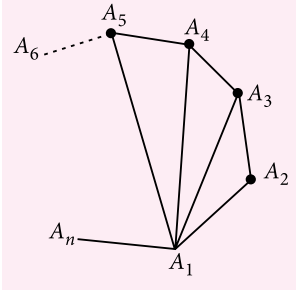
3. Let $\sin x + \sin y = a$ and $\cos x + \cos y = b$, show that $\tan \frac{x}{2}$ and $\tan \frac{y}{2}$ are two roots of the equation:

$(a^2 + b^2 + 2b)t^2 - 4at + (a^2 + b^2 - 2b) = 0$

4. In a triangle ABC, angle A is twice the angle B. Show that $a^2 = b(b + c)$, where a, b and c are the sides opposite to angle A, B and C respectively.

5. A, B, C, D are four points on a circle with radius R such that AC is perpendicular to BD and meets BD at E. Prove that : $EA^2 + EB^2 + EC^2 + ED^2 = 4R^2$.

6. Suppose $A_1A_2A_3 \ldots\ldots\ldots A_n$ is an n-sided regular polygon such that



$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$. Determine the number of sides of the polygon.

7. Find all positive integers a, b for which the number $\frac{\sqrt{2}+\sqrt{a}}{\sqrt{3}+\sqrt{b}}$ is a rational number.

8. If a, b, c are positive real numbers, prove that :

$$\frac{\sqrt{a+b+c}+\sqrt{a}}{b+c} + \frac{\sqrt{a+b+c}}{c+a} + \frac{\sqrt{a+b+c}+\sqrt{c}}{a+b} \geq \frac{9+3\sqrt{3}}{2\sqrt{a+b+c}}$$

9. Find integers a and b such that $x^2 - x - 1$ divides $ax^{17} + bx^{16} + 1 = 0$.

10. Consider the equation $x^4 - 18x^3 + kx^2 + 174x - 2015 = 0$. If the product of two of the four roots of the equation is -31, find the value of k.

SOLUTIONS

1. **Case 1 :** When n is even number $\Rightarrow n = 2k$ for some $k \in I$.

Now, $n^6 + 3n^5 - 15n^3 + 4n^2 + 12n + 3$ becomes

$64k^6 + 96k^5 - 80k^4 - 120k^3 + 16k^2 + 24k + 3$

$\Rightarrow$ $4(\lambda) + 3$, where $\lambda \in I$.

Now an odd perfect square is always of the form $4\lambda + 1$ ($\because$ $(2m + 1)^2 = 4(m^2 + m) + 1$), where $m \in I$.

The given expression can't be perfect square for any even integer.

Case 2 : When n is odd number $\Rightarrow n = 2k' + 1$ for some $k' \in I$.

Now, $n^6 + 3n^5 - 5n^4 - 15n^3 + 4n^2 + 12n + 3$ becomes $(2k' + 1)^6 + 3(2k' + 1)^5 - 5(2k' + 1)^4 - 15(2k' + 1)^3 + 4(2k' + 1)^2 + 12(2k' + 1) + 3$

Now, $(2k' + 1)^6 \equiv (12k' + 1) \pmod 4 \equiv 1 \pmod 4$

$3(2k' + 1)^5 \equiv (30k' + 3) \pmod 4 \equiv (2k' + 3) \pmod 4$

$5(2k' + 1)^4 \equiv (40k' + 5) \pmod 4 \equiv 1 \pmod 4$

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$15(2k'+1)^3 \equiv (90k'+15) \pmod 4 \equiv (2k'+3) \pmod 4$

$4(2k'+1)^2 \equiv 0 \pmod 4$

$12(2k'+1) \equiv 0 \pmod 4$

or, $n^6 + 3n^5 - 5n^4 - 15n^3 + 4n^2 + 12n + 3 \equiv (1 + 2k' + 3 + 1 + 2k' + 3 + 3) \pmod 4$

$\equiv (4(k'+2)+3) \pmod 4$

We have $n^6 + 3n^5 - 5n^4 - 15n^3 + 4n^2 + 12n + 3 = 4\lambda' + 3$ (where $\lambda' \in N$)

Again the expression is not a perfect square for all odd integers.

Consequently, the expression is not a perfect square for any integer.

2. Two normal dice are thrown, possible outcomes are

$$\left.\begin{array}{l}\{(1,1),(1,2),(1,3),.....,(1,6),\\ (2,1),(2,2),(2,3),.....,(2,6),\\ (3,1),(3,2),(3,3),.....,(3,6),\\ .....\\ (6,1),(6,2),(6,3),.....,(6,6)\}\end{array}\right\} \text{total 36 outcomes}$$

Now, E = sum of numbers on dice faces.

$= \{2, 3, 4, ..., 12\}$

$n(E) = 11.$

Loading the dice means tampering with the number on the faces of dice so that one result is favoured or dismayed over other.

For a normal throw

$$P(n)_{n\in E} = \begin{cases} \dfrac{n-1}{36}, & 2 \le n \le 7 \\ \dfrac{12-(n-1)}{36}, & 7 < n \le 12 \end{cases}$$

If all members of E are equally likely, then,

$$P(n)_{\forall n\in E} = \frac{k}{36} \text{ for some } k \in N.$$

Also, outcomes of E are mutually exclusive and mutually exhaustive. So,

$$\sum_{\forall n\in E} P(n) = 1 \Rightarrow 11 \times \frac{k}{36} = 1 \Rightarrow 11 \times k = 36$$

Now, k can't be a natural number

$\Rightarrow$ such loading is not possible.

3. $\sin x + \sin y = a$

$$\Rightarrow 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = a \quad \ldots(i)$$

$\cos x + \cos y = b$

$$\Rightarrow 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = b \quad \ldots(ii)$$

to prove $\tan\frac{x}{2}$ and $\tan\frac{y}{2}$ are roots of the equation

$(a^2 + b^2 + 2b)t^2 - 4at + (a^2 + b^2 - 2b) = 0$

Let the roots of the equation be α and β.

Sum of roots $\alpha + \beta = \dfrac{4a}{a^2+b^2+2b}$ and product of roots $\alpha\beta = \dfrac{a^2+b^2-2b}{a^2+b^2+2b}$

Now, $a^2 + b^2 = \sin^2 x + \sin^2 y + 2\sin x \sin y + \cos^2 x + \cos^2 y + 2\cos x \cos y.$

$= 2(1 + \sin x \sin y + \cos x \cos y)$

$= 2(1 + \cos(x - y)) = 4\cos^2\left(\dfrac{x-y}{2}\right)$

Clearly, $a^2 + b^2 + 2b =$

$$4\cos\left(\frac{x-y}{2}\right)\left[\cos\left(\frac{x-y}{2}\right) + \cos\left(\frac{x+y}{2}\right)\right]$$

$$= 4\cos\left(\frac{x-y}{2}\right)\left(2\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\right) \quad \ldots.(iii)$$

and $a^2 + b^2 - 2b$

$$= 4\cos\left(\frac{x-y}{2}\right)\left[\cos\left(\frac{x-y}{2}\right) - \cos\left(\frac{x+y}{2}\right)\right]$$

$$= 4\cos\left(\frac{x-y}{2}\right)\left[2\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\right] \quad \ldots(iv)$$

Now, sum of roots $= \dfrac{4a}{a^2+b^2+2b}$

$$= \frac{4 \times 2\sin\left(\frac{x+y}{2}\right)}{8\sin\frac{x}{2}.\sin\left(\frac{y}{2}\right)} = \tan\frac{y}{2} + \tan\frac{x}{2}$$

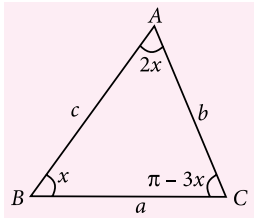
& product of roots $= \dfrac{a^2+b^2-2b}{a^2+b^2+2b} = \tan\frac{x}{2}.\tan\frac{y}{2}$

Thus, roots are $\tan\left(\frac{x}{2}\right)$ and $\tan\left(\frac{y}{2}\right)$.

4. In ΔABC, $\angle A$ is twice the $\angle B$.

Let $\angle A = 2x$, so $\angle B = x$.

$\Rightarrow \angle C = \pi - 3x$



By sine law, we have

$$\frac{a}{\sin 2x} = \frac{b}{\sin x} = \frac{c}{\sin(\pi - 3x)}$$

$$\Rightarrow \frac{a}{2\sin x\cos x} = \frac{b}{\sin x} = \frac{c}{\sin 3x}$$

$$\Rightarrow \frac{a}{2\sin x\cos x} = \frac{b}{\sin x} = \frac{c}{\sin x(3-4\sin^2 x)}$$

$$\Rightarrow \frac{a}{2\cos x} = b = \frac{c}{4\cos^2 x - 1} \quad \because \sin x \neq 0 \text{ as } x \neq n\pi$$

Now, we have $\cos x = \frac{a}{2b} \Rightarrow \cos^2 x = \frac{a^2}{4b^2}$

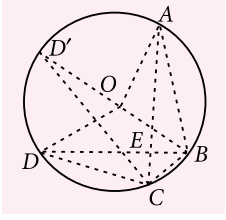
and $4\cos^2 x - 1 = \frac{c}{b} \Rightarrow \cos^2 x = \left(\frac{\frac{c}{b}+1}{4}\right)$

On comparing, we get $\frac{a^2}{4b^2} = \frac{b+c}{4b}$

$\Rightarrow a^2 = b(b + c)$.

5. A, B, C and D are concyclic points with O as centre of circle.

$AC \perp BD$ and AC meets BD at E.



Construction : Extend BO to meet the circle at D'.

In ΔDEC and ΔAEB

We have $\angle DEC = \angle AEB = 90°$

$\angle CDB = \angle CAB$

We have $\Delta DEC \sim \Delta AEB$ (by AA similarity)

Also, $\frac{ED}{EC} = \frac{EA}{EB}$

or $\frac{ED^2}{EC^2} = \frac{EA^2}{EB^2} \Rightarrow \frac{ED^2 + EC^2}{EC^2} = \frac{EA^2 + EB^2}{EB^2}$

(applying componendo)

Now, $ED^2 + EC^2 = \frac{EC^2}{EB^2}(EA^2 + EB^2)$ …(i)

L.H.S. : $EA^2 + EB^2 + EC^2 + ED^2$

$$= EA^2 + EB^2 + \frac{EC^2}{EB^2}(EA^2 + EB^2)$$

$$= \frac{(EA^2 + EB^2)(EB^2 + EC^2)}{EB^2}$$

$$= \frac{AB^2 \cdot BC^2}{EB^2} \quad \ldots(ii)$$

Now, in ΔEAB and $\Delta CD'B$, we have

$\angle BD'C = \angle BAC$

and $\angle D'CB = \angle AEB = 90°$

so $\Delta AEB \sim \Delta D'CB$ (by AA similarity)

or $\frac{BA}{BE} = \frac{BD'}{BC}$

$\Rightarrow \frac{BA^2}{BE^2} BC^2 = BD'^2$ …(iii)

Comparing (ii) & (iii), we get

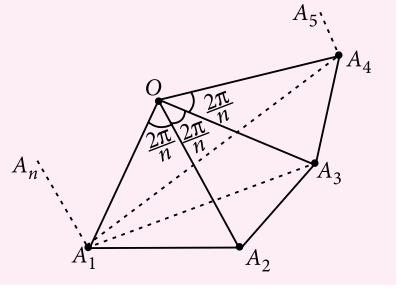
$EA^2 + EB^2 + EC^2 + ED^2$ = (Diameter)^2

= 4 × (radius)^2

6. We have, n-sided regular polygon whose vertices are $A_1, A_2, A_3, \ldots\ldots, A_n$.

Clearly, $\angle A_1OA_2 = \frac{2\pi}{n}$ and let $OA_1 = r_1$.

In ΔA_1OA_2, we have $\cos\frac{2\pi}{n} = \frac{2r^2 - (A_1A_2)^2}{2r^2}$



$$\Rightarrow (A_1A_2)^2 = 2r^2\left(1 - \cos\frac{2\pi}{n}\right) \Rightarrow A_1A_2 = 2r\sin\left(\frac{\pi}{n}\right)$$

Similarly, in ΔA_1OA_3 we have, $A_1A_3 = 2r\sin\left(\frac{2\pi}{n}\right)$

and in ΔA_1OA_4, we have, $A_1A_4 = 2r\sin\left(\frac{3\pi}{n}\right)$

We have, $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$

$$\Rightarrow \frac{1}{2r\sin\left(\frac{\pi}{n}\right)}=\frac{1}{2r\sin\left(\frac{2\pi}{n}\right)}+\frac{1}{2r\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{\pi}{n}\right)}-\frac{1}{\sin\left(\frac{3\pi}{n}\right)}=\frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \sin\left(\frac{3\pi}{n}\right)-\sin\left(\frac{\pi}{n}\right)=\frac{\sin\left(\frac{\pi}{n}\right).\sin\left(\frac{3\pi}{n}\right)}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow 2\sin\left(\frac{\pi}{n}\right).\cos\left(\frac{2\pi}{n}\right).\sin\left(\frac{2\pi}{n}\right)=\sin\left(\frac{\pi}{n}\right).\sin\left(\frac{3\pi}{n}\right)$$

$\because \sin\left(\frac{\pi}{n}\right)\neq 0 \Rightarrow \sin\left(\frac{4\pi}{n}\right)=\sin\left(\frac{3\pi}{n}\right)$

$\Rightarrow \frac{4\pi}{n}=\pi-\frac{3\pi}{n}$ or $\frac{4\pi}{n}=2\pi+\frac{3\pi}{n}$.....

$\Rightarrow \frac{7\pi}{n}=\pi$ or $\frac{\pi}{n}=2\pi$.....

$\Rightarrow n=7$ or $n=\frac{1}{2}$.....

$\because$ n is an integer.

$\Rightarrow$ Number of sides = 7.

7. Let $N=\frac{\sqrt{2}+\sqrt{a}}{\sqrt{3}+\sqrt{b}}=\frac{p}{q}$, where p, q are non zero integers.

or $N=\frac{\sqrt{2}+\sqrt{a}}{\sqrt{3}+\sqrt{b}}\times\frac{\sqrt{3}-\sqrt{b}}{\sqrt{3}-\sqrt{b}}$ (assuming $b\neq 3$)

or $N=\frac{\sqrt{6}-\sqrt{2b}+\sqrt{3a}-\sqrt{ab}}{3-b}$

Now $\sqrt{6}+\sqrt{3a}-\sqrt{2b}-\sqrt{ab}$ is a non–zero integer.

Case1 : $2b = 6 \Rightarrow b = 3$ (not possible)

Case 2 : $ab = 6$ and $\sqrt{3a}-\sqrt{2b}$ is an integer

$\because$ a and b are positive integers.

$\therefore$ Possible pairs of (a, b) are (1, 6), (3, 2), (6, 1) (we can't take $b = 3$)

Now, for (a, b) as (1, 6), we get

$$\sqrt{3a}-\sqrt{2b}=-\sqrt{3}$$

For (a, b) as (3, 2), we get

$$\sqrt{3a}-\sqrt{2b}=1$$

For (a, b) as (6, 1), we get

$$\sqrt{3a}-\sqrt{2b}=2\sqrt{2}$$

Possible solution is (3, 2)

Case 3 : $\sqrt{6}=\sqrt{2b}+\sqrt{ab}$ and $a = 3k_1^2$, for some $k_1 \in N$.

$\Rightarrow$ $6 = 2b + ab + 2b\sqrt{2a}$

R.H.S. must be integer $\Rightarrow a = 2k_2^2$ for some $k_2 \in I$.

We are contradicting our earlier specification that $a = 3k_1^2$ & also $a = 2k_2^2$

Thus, no solution exists.

Possible solution of (a, b) is (3, 2).

8. $\frac{\sqrt{a+b+c}+\sqrt{a}}{b+c}=\frac{1}{\left(\sqrt{a+b+c}-\sqrt{a}\right)}$

Multiplying numerator and denominator by $\left(\sqrt{a+b+c}-\sqrt{a}\right)$

Similarly for other two ratios, consequently

$$\text{L.H.S.}=\frac{1}{\sqrt{a+b+c}-\sqrt{a}}+\frac{1}{\sqrt{a+b+c}-\sqrt{b}}+\frac{1}{\sqrt{a+b+c}-\sqrt{c}}$$

For three positive real numbers, we have A.M. $\geq$ H.M.

$$\Rightarrow \frac{\left(\frac{1}{\sqrt{a+b+c}-\sqrt{a}}+\frac{1}{\sqrt{a+b+c}-\sqrt{b}}+\frac{1}{\sqrt{a+b+c}-\sqrt{c}}\right)}{3}=\alpha(\text{let})$$

$$\geq\left(\frac{3}{3\sqrt{a+b+c}-\sqrt{a}-\sqrt{b}-\sqrt{c}}\right)$$

$$\Rightarrow \alpha\geq\frac{9}{\sqrt{a+b+c}\left(3-\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{a+b+c}}\right)}$$

Let $\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{a+b+c}}=\beta$

Also, $\sqrt{a}+\sqrt{b}+\sqrt{c}\geq\sqrt{3}\sqrt{a+b+c}$

$\because \frac{a_1^m+a_2^m+....+a_n^m}{n}\geq\left(\frac{a_1+a_2+a_3+...+a_n}{n}\right)^m$,

$a_i > 0, i \in N$

We have, $\beta\geq\sqrt{3}$ or $3-\beta\leq 3-\sqrt{3}$

or $\frac{1}{3-\beta} \geq \frac{1}{3-\sqrt{3}}$

or $\frac{9}{\sqrt{a+b+c}(3-\beta)} \geq \frac{9}{(3-\sqrt{3})\sqrt{a+b+c}}$

Also, $\alpha \geq \frac{9}{\sqrt{a+b+c}(3-\beta)} \geq \frac{3(3+\sqrt{3})}{2\sqrt{a+b+c}}$

$\Rightarrow \frac{1}{\sqrt{a+b+c}-\sqrt{a}} + \frac{1}{\sqrt{a+b+c}-\sqrt{b}} + \frac{1}{\sqrt{a+b+c}-\sqrt{c}} \geq \frac{9+3\sqrt{3}}{2\sqrt{a+b+c}}$

Hence proved.

9. Let α be roots of $x^2 - x - 1$. Then $\alpha^2 - \alpha - 1 = 0$.

or $\alpha^2 = \alpha + 1$ ...(i)

Now, as $(x^2 - x - 1)$ divides $ax^{17} + bx^{16} + 1 = 0$

$\Rightarrow$ α is a root of $ax^{17} + bx^{16} + 1 = 0$

$\Rightarrow a\alpha^{17} + b\alpha^{16} + 1 = 0 \Rightarrow \alpha^{16}(a\alpha + b) + 1 = 0$

$\Rightarrow (\alpha^2)^8 (a\alpha + b) + 1 = 0$

$\Rightarrow (\alpha + 1)^8 (a\alpha + b) + 1 = 0$ [putting $\alpha^2 = \alpha + 1$]

$\Rightarrow (\alpha^2 + 2\alpha + 1)^4 (a\alpha + b) + 1 = 0$

$\Rightarrow (3\alpha + 2)^4 (a\alpha + b) + 1 = 0$ [putting $\alpha^2 = \alpha + 1$]

$\Rightarrow (9\alpha^2 + 12\alpha + 4)^2 (a\alpha + b) + 1 = 0$

$\Rightarrow (21\alpha + 13)^2 (a\alpha + b) + 1 = 0$ [putting $\alpha^2 = \alpha + 1$]

$\Rightarrow (441\alpha^2 + 546\alpha + 169)(a\alpha + b) + 1 = 0$

$\Rightarrow (987\alpha + 610)(a\alpha + b) + 1 = 0$ [putting $\alpha^2 = \alpha + 1$]

$\Rightarrow 987a\alpha^2 + \alpha(987b + 610a) + 610b + 1 = 0$

$\Rightarrow \alpha(987b + 610a + 987a) + 610b + 1 = 0$

$\Rightarrow (\alpha \times k_1) + k_2 = 0$, where $k_1, k_2 \in I$

or $(1597a + 987b)\alpha + 987a + 610b + 1 = 0$

Also, $\alpha = \frac{1 \pm \sqrt{5}}{2}$

or, $(1597a + 987b) \times \left(\frac{\pm\sqrt{5}}{2}\right) + 2584a + 1597b + 1 = 0$

or, $1597a + 987b = 0$ ...(i)

and $2584a + 1597b + 1 = 0$ ...(ii)

We have $a = -\frac{987}{1597}b$, putting in (ii), we get

$b\left[1597 - \frac{987 \times 584}{1597}\right] + 1 = 0$

$\Rightarrow b = -1597 \Rightarrow a = 987$

10. $x^4 - 18x^3 + kx^2 + 174x - 2015 = 0$

Let the roots be a, b, c and d; and $ab = -31$

We have $a + b + c + d = 18$ ...(i)

$ab + ac + ad + bc + bd + cd = k$ ...(ii)

$abc + abd + acd + cbd = -174$ ...(iii)

$abcd = -2015$ ...(iv)

From (iv), we have $cd = 65$

From (iii), we have $ab(c + d) + cd(b + a) = -174$

$\Rightarrow -31(c + d) + 65(a + b) = -174$

Let $(c + d) = p$ and $a + b = q$

We have, $p + q = 18$ and $65q - 31p = -174$

$\Rightarrow p = 4$ and $q = 14$

$\Rightarrow c + d = 4$ & $a + b = 14$

From (ii),

$ab + a(c + d) + (c + d) + cd = k$

$\Rightarrow -31 + (c + d)(a + b) + 65 = k$

$\Rightarrow 34 + 56 = k \Rightarrow k = 90$

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MATHS MUSING

SOLUTION SET-144

1. (b) : $2n + 1 = (n + 1)^2 - n^2$

$$n^3 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{n(n-1)}{2}\right)^2$$

$\therefore$ m contains all odd numbers and the even numbers $2^3, 4^3, 6^3, 8^3, 10^3, 12^3$.

$\therefore$ $m = 1007 + 6 = 1013$ with digit sum 5.

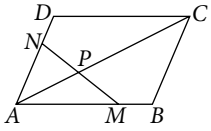
2. (c): $\overrightarrow{AB} = \vec{b}, \overrightarrow{AD} = \vec{d}$

The line AC is

$\vec{r} = p(\vec{b} + \vec{d})$

The line MN is

$\vec{r} = m\vec{b} + q(n\vec{d} - m\vec{b})$



P is a common point $\Rightarrow p = m(1 - q) = qn$

$\therefore$ $p = \dfrac{mn}{m+n}$

3. (c): $z^{18} = 1, \omega^{48} = 1$

$(z\,\omega)^n = 1, z^n\omega^n = 1 = 1 \times 1$

$\therefore$ n is divisible by 18 and 48. L.C.M = 144 with digit sum 9.

4. (a): $AA\ NN\ RR\ DE$

$$2+2+1 \Rightarrow \binom{3}{2}\binom{3}{1}\frac{5!}{2!2!} = 270$$

$$2+1+1+1 \Rightarrow \binom{3}{1}\binom{4}{3}\frac{5!}{2!} = 720$$

$1 + 1 + 1 + 1 + 1 \Rightarrow 5! = 120$

$\therefore$ $n = 270 + 720 + 120 = 1110$, digit sum 3

5. (b) : $\dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} = dx$

$\therefore$ $\sqrt{x^2 + y^2} = x + c,\ y^2 = c^2 + 2cx$, parabolas

6. (a,b,d) : $A_{4n} + A_{4n+1} = (4x)^2$

$A_{4n+2} + A_{4n+3} = -(4n + 2)^2$

$\therefore$ $N = -2^2 + 4^2 - 6^2 + 8^2 - \ldots\ldots + 2012^2 - 2014^2$

$= -4[1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots - 1006^2 + 1007^2]$

$= -4[-503 \cdot 1007 + 1007^2]$

$= -4 \cdot 1007 \cdot 504 = -2^5 \cdot 3^2 \cdot 7 \cdot 19 \cdot 53$

7. (a): Let BC subtend angle 2α at the centre. Thus AB subtend angle $2\pi - 10$ at the centre.

$2r\sin\alpha = 81,\ 2r\sin 5\alpha = 31$

$$\sin 5\alpha = \frac{31}{81}\sin\alpha \Rightarrow 16\sin^4\alpha - 20\sin^2\alpha$$

Solving, $\sin\alpha = \dfrac{\sqrt{11}}{6}$

$$r = \frac{81}{2\sin\alpha} = \frac{243}{\sqrt{11}}$$

8. (c): $BE = 2r\sin 3\alpha$

$$= 2 - \frac{243}{\sqrt{11}}(3\sin\alpha - 4\sin^3\alpha) = 144$$

9. (3): $\dfrac{5^{a_{n+1}}}{5^{a_n}} = \dfrac{3n+5}{3n+2} = \dfrac{3(n+1)+2}{3n+2}$

$\therefore$ $5^{a_n} = \alpha(3n+2),\ a_1 = 1 \Rightarrow \alpha = 1$

$\therefore$ $5^{a_n} = 3n+2,\ a_n = \log_5(3n+2)$

$a_{41} = 3$

10. (a) : P. $f(x) = \dfrac{1}{\pi^2}(\cos^{-1}x + \sin^{-1}x) \times (\cos^{-1}x - \sin^{-1}x)$

$$= \frac{1}{2\pi}(\cos^{-1}x - \sin^{-1}x), \text{ decreasing function}$$

$\therefore$ $b - a = f(-1) - f(1) = \dfrac{3}{4} + \dfrac{1}{4} = 1$

Q. $f(x) = \dfrac{2}{\pi^2}\left[\left(\sin^{-1}x - \dfrac{\pi}{4}\right)^2 + \dfrac{\pi^2}{16}\right]$

$b - a = \dfrac{5}{4} - \dfrac{1}{8} = \dfrac{9}{8}$

R. $f(x)$ decreases $b - a = f(-1) - f(1)$

$= \dfrac{9}{8} + \dfrac{1}{8} = \dfrac{5}{4}$

S. $f(x) = \dfrac{3}{2\pi^2}\left[\left(\sin^{-1}x - \dfrac{\pi}{4}\right)^2 + \dfrac{\pi^2}{48}\right]$

$b - a = \dfrac{7}{8} - \dfrac{1}{32} = \dfrac{27}{32}$

Solution Sender of Maths Musing

SET-144

1.	N. Jayanthi	:	Hyderabad
2.	Khokon Kumar Nandi	:	W.B.
3.	Gouri Sankar Adhikari Mayta	:	W.B.

YOU ASKED WE ANSWERED

Do you have a question that you just can't get answered?

Use the vast expertise of our mtg team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

Q1. Prove that the equation $\sin^6 x + \cos^6 x = a$ is solvable if $\frac{1}{4} \le a \le 1$. *- Sujatha, Thrissur*

Ans. We have, $\sin^6 x + \cos^6 x = a$

$\Rightarrow (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x (\sin^2 x + \cos^2 x) = a$

$\Rightarrow 1 - 3\sin^2 x \cos^2 x = a \Rightarrow 1 - \frac{3}{4}\sin^2 2x = a$

$\Rightarrow 1 - \frac{3}{4}\left(\frac{1-\cos 4x}{2}\right) = a$

$\Rightarrow 1 - \frac{3}{8} + \frac{3}{8}\cos 4x = a$

$\Rightarrow \frac{5}{8} + \frac{3}{8}\cos 4x = a$ ...(i)

But $-1 \le \cos 4x \le 1 \Rightarrow -\frac{3}{8} \le \frac{3}{8}\cos 4x \le \frac{3}{8}$

$\Rightarrow \frac{5}{8} - \frac{3}{8} \le \frac{5}{8} + \frac{3}{8}\cos 4x \le \frac{3}{8} + \frac{5}{8}$

$\Rightarrow \frac{1}{4} \le a \le 1$ (by (i))

Q2. Find the equation of the plane passing through the point (–1, 2, 1) and perpendicular to the line joining the points (–3, 1, 2) and (2, 3, 4). Find also the perpendicular distance of the origin from this plane. *- Mahendra, Amritsar*

Ans. The required plane passes through the point $(-1, 2, 1)$ having position vector $\vec{a} = -\hat{i} + 2\hat{j} + \hat{k}$ and is perpendicular to the line joining the points $A(-3, 1, 2)$ and $B(2, 3, 4)$.

$\therefore$ A vector normal to the plane is given by

$\vec{n} = \overrightarrow{AB} = (2\hat{i} + 3\hat{j} + 4\hat{k}) - (-3\hat{i} + \hat{j} + 2\hat{k})$

$= 5\hat{i} + 2\hat{j} + 2\hat{k}$

Vector equation of a plane passing through point having position vector $\vec{a}$ and normal to vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \Rightarrow \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\therefore$ Equation of the required plane is

$\vec{r} \cdot (5\hat{i} + 2\hat{j} + 2\hat{k}) = (-\hat{i} + 2\hat{j} + \hat{k}) \cdot (5\hat{i} + 2\hat{j} + 2\hat{k})$

$\Rightarrow \vec{r} \cdot (5\hat{i} + 2\hat{j} + 2\hat{k}) = -5 + 4 + 2$

$\Rightarrow \vec{r} \cdot (5\hat{i} + 2\hat{j} + 2\hat{k}) = 1$

We have $|\vec{n}| = \sqrt{5^2 + 2^2 + 2^2} = \sqrt{33}$

In normal form

$\vec{r} \cdot \left(\frac{5}{\sqrt{33}}\hat{i} + \frac{2}{\sqrt{33}}\hat{j} + \frac{2}{\sqrt{33}}\hat{k}\right) = \frac{1}{\sqrt{33}}$

So, the perpendicular distance of the origin from the plane is $\frac{1}{\sqrt{33}}$.

Q3. On each evening a boy either watches Doordarshan channel or Ten sports. The probability that he watches Ten sports is $\frac{4}{5}$. If he watches Doordarshan, there is a chance of $\frac{3}{4}$ that he will fall asleep, while it is $\frac{1}{4}$ when he watches Ten sports. On one day, the boy is found to be asleep. Find the probability that the boy watched Doordarshan.

- Anjali Jha, Patna

Ans. Let E_1 and E_2 be the events of the boy watching Doordarshan and Ten sports, respectively. It is given that

$P(E_1) = \frac{1}{5}$ and $P(E_2) = \frac{4}{5}$

Let E be the event of the boy falls asleep. Again by hypothesis

$P\left(\frac{E}{E_1}\right) = \frac{3}{4}$ and $P\left(\frac{E}{E_2}\right) = \frac{1}{4}$

Now $E = E \cap (E_1 \cup E_2) = (E_1 \cap E) \cup (E_2 \cap E)$

so that $P(E) = P(E_1)P\left(\frac{E}{E_1}\right) + P(E_2)P\left(\frac{E}{E_2}\right)$

By Bayes' theorem

$$P(E_1/E) = \frac{P(E_1)\,P(E/E_1)}{P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2)}$$

$$= \frac{(1/5) \times (3/4)}{(1/5) \times (3/4) + (4/5) \times (1/4)} = \frac{3}{7}$$

concept BOOSTERS

Class XI

Circles

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

STANDARD RESULTS

EQUATION OF A CIRCLE IN VARIOUS FORM

- The equation of circle with centre (h, k) and radius 'r' is $(x - h)^2 + (y - k)^2 = r^2$.
- The general equation of a circle is
  $x^2 + y^2 + 2gx + 2fy + c = 0$
  with centre $(-g, -f)$ & radius $= \sqrt{g^2 + f^2 - c}$.
  Remember that every second degree equation in x and y in which coefficient of x^2 = coefficient of y^2 & there is no xy term always represents a circle.
  If $g^2 + f^2 - c > 0 \Rightarrow$ real circle.
  $g^2 + f^2 - c = 0 \Rightarrow$ point circle.
  $g^2 + f^2 - c < 0 \Rightarrow$ imaginary circle.
  Note that the general equation of a circle contains three arbitrary constants, g, f and c which corresponds to the fact that a unique circle passes through three non-collinear points.
- The equation of circle with (x_1, y_1) and (x_2, y_2) as end points of its diameter is,
  $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
  Note that this will be the circle of least radius passing through (x_1, y_1) and (x_2, y_2).

INTERCEPTS MADE BY A CIRCLE ON THE AXES

The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with the coordinate axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively.

Note :

If $g^2 - c > 0 \Rightarrow$ circle cuts the x-axis at two distinct points.

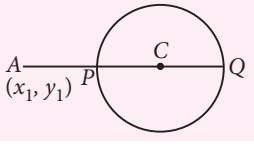
If $g^2 = c \Rightarrow$ circle touches the x-axis.

If $g^2 < c \Rightarrow$ circle lies completely above or below the x-axis.

POSITION OF A POINT W.R.T. A CIRCLE

The point (x_1, y_1) lies inside, on or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < = \text{or} > 0$ respectively.

Note : The greatest and the least distance of a point A from a circle with centre C and radius r is $AC + r$ and $AC - r$ respectively.



LINE AND CIRCLE

Let $L = 0$ be a line and $S = 0$ be a circle. If r is the radius of the circle and p is the length of the perpendicular from the centre on the line, then

- $p > r \Leftrightarrow$ the line does not meet the circle *i.e.* the line passes outside the circle.
- $p = r \Leftrightarrow$ the line touches the circle.
- $p < r \Leftrightarrow$ the line is a secant of the circle.
- $p = 0 \Rightarrow$ the line is a diameter of the circle.

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

PARAMETRIC EQUATIONS OF A CIRCLE

The parametric equations of a circle $(x-h)^2+(y-k)^2=r^2$ are,

$x = h + r\cos\theta$; $y = k + r\sin\theta$; $-\pi < \theta \le \pi$ where (h, k) is the centre, r is the radius and θ is a parameter.

Note that equation of a straight line joining two points α & β on the circle $x^2 + y^2 = a^2$ is

$$x\cos\left(\frac{\alpha+\beta}{2}\right)+y\sin\left(\frac{\alpha+\beta}{2}\right)=a\cos\left(\frac{\alpha-\beta}{2}\right)$$

TANGENT & NORMAL

- The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is, $xx_1 + yy_1 = a^2$. Hence equation of a tangent at $(a\cos\alpha, a\sin\alpha)$ is; $x\cos\alpha + y\sin\alpha = a$. The point of intersection of the tangents at the points $P(\alpha)$and $Q(\beta)$ is

$$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)},\ \frac{a\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

- The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
- $y = mx + c$ is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ and the point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$.
- If a line is normal / orthogonal to a circle, then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

FAMILY OF CIRCLES

- Equation of circle circumscribing a triangle whose sides are $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ & coefficient of x^2 = coefficient of y^2.
- Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0, L_2 = 0, L_3 = 0$ and $L_4 = 0$ is $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.
- The equation of the family of circles passing through the point of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is, $S_1 + KS_2 = 0$ $(K \ne -1)$.
- The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.
- The equation of a family of circles passing through two given points (x_1, y_1) and (x_2, y_2) is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2)$

$$+K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \text{where } K \text{ is a parameter.}$$

- The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

  In case, if the line through (x_1, y_1) is parallel to y-axis then the equation of the family of circles touching it at (x_1, y_1) becomes $(x - x_1)^2 + (y - y_1)^2 + K(x - x_1) = 0$.

  Also, if line is parallel to x-axis the equation of the family of circles touching it at (x_1, y_1) becomes $(x - x_1)^2 + (y - y_1)^2 + K(y - y_1) = 0$.

LENGTH OF A TANGENT AND POWER OF A POINT

The length of a tangent from an external point (x_1, y_1) to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$$L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$$

Square of length of the tangent from a point P is also called the power of point w.r.t. a circle. Power of a point remains constant w.r.t. a circle.

Note that power of a point P is positive, negative or zero according as the point 'P' is outside, inside or on the circle respectively.

DIRECTOR CIRCLE

The locus of the point of intersection of two perpendicular tangents is called the director circle of the given circle. The director circle of

a circle is the concentric circle having radius equal to $\sqrt{2}$ times the original circle.

EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT

- The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $y - y_1 = -\dfrac{x_1 + g}{y_1 + f}(x - x_1)$.

  This on simplication can be put in the form $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

  Note that the shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

CHORD OF CONTACT

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact of T_1T_2 is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

REMEMBER

- Equation of the circle circumscribing the ΔPT_1T_2 is $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.
- The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $SS_1 = T^2$.
  where, $S \equiv x^2 + y^2 + 2gx + 2fy + c$;
  $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
  $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.
- Area of the triangle formed by the pair of the tangents & its chord of contact $= \dfrac{RL^3}{R^2 + L^2}$

  where R is the radius of the circle & L is the length of the tangent from (x_1, y_1) on $S = 0$
- Chord of contact exists only if the point 'P' is not inside.
- Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$.
- Angle between the pair of tangents from $(x_1, y_1) = \tan^{-1}\left(\dfrac{2RL}{L^2 - R^2}\right)$ where R = radius; L = length of tangent.

POLE AND POLAR

- If through a point P in the plane of the circle, two straight lines are drawn to meet the circle at points Q and R, the locus of the point of intersection of the tangents at Q and R is called the polar of the point P; also P is called the pole of the polar.
- The equation of the polar of a point $P(x_1, y_1)$ w.r.t. the circle $x^2 + y^2 = a^2$ is given by $xx_1 + yy_1 = a^2$, and if the circle is general, then the equation of the polar becomes $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
  Note that if the point (x_1, y_1) be on the circle, then the chord of contact, tangent and polar will be represented by the same equation.
- Pole of a given line $Ax + By + C = 0$ w.r.t. any circle $x^2 + y^2 = a^2$ is $\left(-\dfrac{Aa^2}{C}, -\dfrac{Ba^2}{C}\right)$.
- If the polar of a point P pass through a point Q, then the polar of Q passes through P.
- Two lines L_1 and L_2 are conjugate of each other if pole of L_1 lies on L_2 and vice-versa. Similarly two points P and Q are said to be conjugate of each other if the polar of P passes through Q and vice-versa.

COMMON TANGENTS TO TWO CIRCLES

- The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.
- Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.
- When they touch each other :

  (a) Externally : There are three common tangents, two direct and one is the tangent at the point of contact .

  (b) Internally : Only one common tangent possible at their point of contact.
- When two circles neither intersect nor touch each other , there are four common tangents, two of them are transverse and the others are direct common tangents.
- When they intersect there are two common tangents, both of them being direct.

- Length of an external common tangent and internal common tangent to the two circles is given by:

$$L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2} \quad \& \quad L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$$

where d = distance between the centres of the two circles, r_1 and r_2 are the radii of the two circles.

RADICAL AXIS & RADICAL CENTRE

- The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of the two circles $S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$
i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.

REMARKS

- Radical axis bisects the common tangent between the two circles.
- The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles.
- A system of circles , every two of which have the same radical axis, is called a co-axial system.
- Pairs of circles which do not have radical axis are concentric.
- If two circles intersect, then the radical axis is the common chord of the two circles.
- If two circles touch each other, then the radical axis is the common tangent of the two circles at the common point of contact.
- Radical axis is always perpendicular to the line joining the centres of the two circles.
- Radical axis need not always pass through the mid point of the line joining the centres of the two circles.

ORTHOGONALITY OF TWO CIRCLES

- Two circles $S_1 = 0$ & $S_2 = 0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

REMARKS

- If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$ are concurrent in a circle which is orthogonal to all the three circles.
- Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles .

PROBLEMS

SECTION-I

Single Correct Answer Type

1. The locus of the mid points of the chords of the circle $x^2 + y^2 - ax - by = 0$ which subtend a right angle at $\left(\frac{a}{2}, \frac{b}{2}\right)$ is

(a) $ax + by = 0$ (b) $ax + by = a^2 + b^2$

(c) $x^2 + y^2 - ax - by + \frac{a^2 + b^2}{8} = 0$

(d) $x^2 + y^2 - ax - by - \frac{a^2 + b^2}{8} = 0$

2. A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is

(a) $8\sqrt{3}$ sq. units (b) $4\sqrt{3}$ sq. units

(c) $16\sqrt{3}$ sq. units (d) none of these

3. In a right triangle ABC, right angled at A, on the leg AC as diameter, a semicircle is described. The chord joining A with the point of intersection D of the hypotenuse and the semicircle, then the length AC equals to

(a) $\frac{AB \cdot AD}{\sqrt{AB^2 + AD^2}}$ (b) $\frac{AB \cdot AD}{AB + AD}$

(c) $\sqrt{AB \cdot AD}$ (d) $\frac{AB \cdot AD}{\sqrt{AB^2 - AD^2}}$

4. The locus of the mid points of the chords of the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ which subtend an angle of $\frac{\pi}{3}$ radians at its circumference is

(a) $(x-2)^2+(y+3)^2=6.25$
(b) $(x+2)^2+(y-3)^2=6.25$
(c) $(x+2)^2+(y-3)^2=18.75$
(d) $(x+2)^2+(y+3)^2=18.75$

5. The angle at which the circles $(x-1)^2+y^2=10$ and $x^2+(y-2)^2=5$ intersect is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

6. The value of 'c' for which the set, $\{(x, y)|x^2+y^2+2x\le 1\}\cap\{(x, y)|\ x-y+c\ge 0\}$ contains only one point in common is

(a) $(-\infty, -1]\cup[3, \infty)$ (b) $\{-1, 3\}$
(c) $\{-3\}$ (d) $\{-1\}$

7. P is a point (a, b) in the first quadrant. If the two circles which pass through P and touch both the coordinate axes cut at right angles, then

(a) $a^2-6ab+b^2=0$ (b) $a^2+2ab-b^2=0$
(c) $a^2-4ab+b^2=0$ (d) $a^2-8ab+b^2=0$

8. Three concentric circles of which the biggest is $x^2+y^2=1$, have their radii in A.P. If the line $y=x+1$ cuts all the circles in real and distinct points. The interval in which the common difference of the A.P. will lie is

(a) $\left(0, \frac{1}{4}\right)$ (b) $\left(0, \frac{1}{2\sqrt{2}}\right)$
(c) $\left(0, \frac{2-\sqrt{2}}{4}\right)$ (d) none of these

9. B and C are fixed points having coordinates (3, 0) and (–3, 0) respectively . If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation

(a) $x^2+y^2=1$ (b) $x^2+y^2=2$
(c) $9(x^2+y^2)=1$ (d) $9(x^2+y^2)=4$

10. If two chords, each bisected by the x-axis can be drawn to the circle, $2(x^2+y^2)-2ax-by=0$ $(a\ne 0, b\ne 0)$ from the point $(a, b/2)$, then

(a) $a^2>8b^2$ (b) $b^2>2a^2$
(c) $a^2>2b^2$ (d) $a^2=2b^2$

11. Tangents are drawn to a unit circle with centre at the origin from a point on the line $2x+y=4$. Then the equation of the locus of the middle point of the chord of contact is

(a) $2(x^2+y^2)=x+y$ (b) $2(x^2+y^2)=x+2y$
(c) $4(x^2+y^2)=2x+y$ (d) none of these

12. The common chord of two intersecting circles C_1 and C_2 can be seen from their centres at the angles of 30° and 60° respectively . If the distance between their centres is equal to $\sqrt{3}+1$, then the radii of C_1 and C_2 are

(a) $\sqrt{3}, 3$ (b) $\sqrt{2}, 2\sqrt{2}$ (c) $\sqrt{2}, 2$ (d) $2\sqrt{2}, 4$

13. If the line $x\cos\theta+y\sin\theta=2$ is the equation of a transverse common tangent to the circles $x^2+y^2=4$ and $x^2+y^2-6\sqrt{3}x-6y+20=0$, then the value of θ is

(a) $5\pi/6$ (b) $2\pi/3$ (c) $\pi/3$ (d) $\pi/6$

14. The locus of the mid-points of the chords of the circle $x^2+y^2-2x-4y-11=0$ which subtend 60° at the centre is

(a) $x^2+y^2-4x-2y-7=0$
(b) $x^2+y^2+4x+2y-7=0$
(c) $x^2+y^2-2x-4y-7=0$
(d) $x^2+y^2+2x+4y+7=0$

15. The locus of the foot of the perpendicular, from the origin to chord of the circle $x^2+y^2-4x-6y-3=0$, which subtend a right angle at the origin is

(a) $x^2+y^2-2x-3y-3=0$
(b) $2x^2+2y^2-4x-6y-9=0$
(c) $3x^2+3y^2-12x-18y-6=0$
(d) $2x^2+2y^2-4x-6y-3=0$

16. Tangents are drawn to $x^2+y^2-1=0$ from any arbitrary point P on the circle $C_1: x^2+y^2-4=0$. These tangent meets the circle "C_1" again at A and B. Locus of point of intersection of tangents drawn to C_1 at A and B is

(a) $x^2+y^2=10$ (b) $x^2+y^2=16$
(c) $x^2+y^2=5$ (d) $x^2+y^2=25$

17. Let PQ and RS be tangents at the extremities of a diameter PR of a circle of radius r such that PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals

(a) $\sqrt{PQ\cdot RS}$ (b) $\frac{PQ+RS}{2}$
(c) $\frac{2PQ\cdot RS}{PQ+RS}$ (d) $\sqrt{\frac{(PQ)^2+(RS)^2}{2}}$

18. Two circles with radii "a" and "b" touch each other externally such that θ is the angle between the direct common tangents ($a > b \geq 2$), then

(a) $\theta = \sin^{-1}\left(\frac{a-b}{a+b}\right)$ (b) $\theta = \sin^{-1}\left(\frac{a+b}{a-b}\right)$

(c) $\theta = 2\sin^{-1}\left(\frac{a-b}{a+b}\right)$ (d) $\theta = 2\sin^{-1}\left(\frac{a+b}{a-b}\right)$

19. P is a point on the circumference of a circle of radius r units. Chords QR are drawn parallel to the tangent at P. The maximum area of the triangle PQR is

(a) $\frac{\sqrt{3}}{4}r^2$ (b) $\frac{3\sqrt{3}}{4}r^2$ (c) $\frac{3}{4}r^2$ (d) $\frac{3\sqrt{3}}{2}r^2$

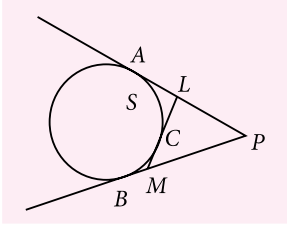
20. A point P lies inside the circles $x^2 + y^2 - 4 = 0$ and $x^2 + y^2 - 8x + 7 = 0$. The point P starts moving under the conditions that its path encloses greatest possible area and it is at a fixed distance from any arbitrarily chosen fixed point in its region. The locus of P is

(a) $4x^2 + 4y^2 - 12x + 1 = 0$
(b) $4x^2 + 4y^2 + 12x - 1 = 0$
(c) $x^2 + y^2 - 3x - 2 = 0$
(d) $x^2 + y^2 - 3x + 2 = 0$

21. Tangents are drawn from (4, 4) to the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ to meet the circle at A and B. The length of the chord AB is

(a) $2\sqrt{3}$ (b) $3\sqrt{2}$ (c) $2\sqrt{6}$ (d) $6\sqrt{2}$

22. PA and PB are tangents to a circle S touching S at points A and B.C is a point on S in between A and B as shown in the figure. LCM is a tangent to S intersecting PA and PB in points L and M respectively. Then the perimeter of the triangle PLM depends on



(a) A, B, C and P (b) P but not on C
(c) P and C only (d) the radius of S only

SECTION-II

Multiple Correct Answer Type

23. ω is a variable circle which passes through the point $A = (2, 3)$ and which touches the x-axis; A' is the point on ω diametrically opposite to A. The curve described by A' is a conic

(a) whose eccentricity is < 1.
(b) whose latus rectum has length 12 units.
(c) which has a vertex at (2, 0).
(d) which has the x-axis as a line of symmetry.

24. If the line $|y| = x - \alpha$; where $\alpha > 0$ does not meet the circle $x^2 + y^2 - 10x + 21 = 0$, then

(a) $\alpha < 5 - 2\sqrt{2}$ (b) $\alpha > 5 + 2\sqrt{2}$
(c) $\alpha \in (5 - 2\sqrt{2}, 5 + 2\sqrt{2})$ (d) $\alpha \in (-1, 1)$

25. The line $y = mx + c$ intersects the circle $x^2 + y^2 = r^2$ at two real and distinct points if

(a) $-r\sqrt{1+m^2} < c \leq 0$ (b) $0 \leq c < r\sqrt{1+m^2}$
(c) $-c\sqrt{1-m^2} < r$ (d) $r < c\sqrt{1+m^2}$

26. In a triangle ABC, let $BC = 13$, $CA = 14$, $AB = 15$. If I is the incentre of the triangle ABC and if the incircle meets the sides AB, BC & CA at the points P, Q and R respectively, then which of the following are correct?

(a) $AP = 8$.
(b) Area of the triangle APR is $\frac{128}{5}$ sq. units
(c) Area of the quadrilateral $APIR$ is 32 sq. units
(d) $AI = 4\sqrt{5}$

27. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to the same circle, then

(a) radius of the circle = 3/4
(b) radius of the circle = 3/2
(c) centre of the circle lies on $12x - 16y + 1 = 0$
(d) centre of the circle lies on $12x - 16y + 31 = 0$

28. A circle $S \equiv 0$ passes through the common points of family of circles $x^2 + y^2 + \lambda x - 4y + 3 = 0$ ($\lambda \in R$) and have minimum area, then

(a) area of $S = 0$ is π sq. units
(b) radius of director circle of $S = 0$ is $\sqrt{2}$ units
(c) length of intercept made by $S \equiv 0$ from x-axis is 1 unit.
(d) All of these

29. $C_1 : x^2 + y^2 = 25$, $C^2 : x^2 + y^2 - 2x - 4y - 7 = 0$ be two circles intersecting at the points A and B

(a) Equation of common chord must be $x + 2y - 9 = 0$

(b) Equation of common chord must be $x + 2y + 7 = 0$.

(c) Tangents at A and B to the circle C_1 intersect at $\left(\frac{25}{9}, \frac{50}{9}\right)$.

(d) Tangents at A and B to the circle C_1 intersect at $(1, 2)$.

30. An equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which makes equal intercepts on the coordinate axes is given by

(a) $x + y = 2\sqrt{2}$ (b) $x - y = 2\sqrt{3}$

(c) $x + y + 2\sqrt{2} = 0$ (d) $x - y + 2\sqrt{3} = 0$

31. If α, β, γ are the parameters of points A, B, C on circle $x^2 + y^2 = a^2$ and if the triangle ABC be equilateral, then

(a) $\Sigma\cos\alpha = 0$ (b) $\Sigma\sin\alpha = 0$

(c) $\Sigma\tan\alpha = 0$ (d) $\Sigma\cot\alpha = 0$

32. Let S_1 and S_2 be two circles passing through (2,3) and touching the coordinate axes and S be the circle having common points of S_1 and S_2 as the centre and radius equal to G.M. of radius of S_1 and S_2, then

(a) $S \equiv 0$ cuts y-axis is but not the x-axis

(b) $S \equiv 0$ cuts $y = x$

(c) radius of director circle of $S \equiv 0$ is $\sqrt{26}$

(d) A.M. of radius of S_1 and S_2 is 5

33. Common tangents are drawn to two circles $x^2 + y^2 = 1$ and $(x - 4)^2 + (y - 4)^2 = 4$. If the intersection points of transverse common tangents with the direct common tangents are A, B, C and D, then

(a) $ABCD$ is a cyclic quadrilateral

(b) equation of circumcircle of ΔABC is $x^2 + y^2 - 4x - 4y = 0$

(c) equation of circumcircle of ΔABC is $x^2 + y^2 - 5x - 5y = 0$

(d) $ABCD$ is a square

34. Let x, y be variables satisfying the equation $x^2 + y^2 + 8x - 10y - 40 = 0$. Let $a = \max\{(x + 2)^2 + (y - 3)^2\}$ and $b = \min\{(x + 2)^2 + (y - 3)^2\}$, then

(a) $a + b = 18$ (b) $a + b = \sqrt{2}$

(c) $a - b = 4\sqrt{2}$ (d) $ab = 73$

SECTION-III

Comprehension Type

Paragraph for Question No. 35 to 37

The centres of the circles touching both the coordinate axes lies on the straight line $x - y = 0$ or $x + y = 0$ according to the quadrant in which the circle lies.

35. The number of circles whose centres are in the first quadrant, touching the axes and passing through a fixed point (h, k) is

(a) always be 1 (b) always be 2

(c) 1 or 2 (d) more than 2

36. If the difference between the radii of the circles passing through (a, b) and touching the axes is $C > 0$ and $ab > 0$, then the least value of $a + b$ is (relative minimum)

(a) C (b) $\sqrt{2}C$ (c) $\frac{C}{\sqrt{2}}$ (d) $2C$

37. If one of the points of intersection of the two circles touching the axes is (p, q), then the length of the common chord is

(a) $|p \pm q|$ (b) $\sqrt{2}\,|p \pm q|$

(c) $2|p \pm q|$ (d) $\sqrt{2}\,|pq|$

Paragraph for Question No. 38 to 40

P is a point on the circle $C_1 : q^2(x^2 + y^2) = a^2p^2$

Q is a point on the circle $C_2 : x^2 + y^2 = a^2$

38. If the coordinates of P are (h, k), then the locus of the point which divides the join of PQ in the ratio $p : q$ is a circle C_3, whose centre is at the point

(a) $\left(\frac{hp}{p+q}, \frac{kq}{p+q}\right)$ (b) $\left(\frac{h}{p+q}, \frac{k}{p+q}\right)$

(c) $\left(\frac{hq}{p+q}, \frac{kq}{p+q}\right)$ (d) $\left(\frac{hp}{p+q}, \frac{kp}{p+q}\right)$

39. Locus of the centre of the circle C_3 as P moves on the circle C_1 is a circle C_4

(a) Non concentric with C_1

(b) Non concentric with C_2

(c) Having radius equal to the radius of C_3

(d) Having area equal to the area of C_1

40. If the point (p, q) lies on the line $y = 2x$, then the $\frac{\text{radius of } C_1}{\text{radius of } C_4}$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 3 (d) $\frac{1}{3}$

Paragraph for Question No. 41 to 43

The line $x + 2y + a = 0$ intersects the circle $x^2 + y^2 - 4 = 0$ at two distinct points A and B. Another line $12x - 6y - 41=0$ intersect the circle $x^2 + y^2 - 4x - 2y + 1 = 0$ at two distinct points C and D.

41. The value of a so that the line $x + 2y + a = 0$ intersect the circle $x^2 + y^2 - 4 = 0$ at two distinct points A and B is

(a) $-2\sqrt{5} < a < 2\sqrt{5}$ (b) $0 < a < 2\sqrt{5}$

(c) $-\sqrt{5} < a < \sqrt{5}$ (d) $0 < a < 2\sqrt{5}$

42. The value of 'a' for which the four points A, B, C and D are concyclic, is

(a) 1 (b) 3 (c) 4 (d) 2

43. The equation of circle passing through the points A, B, C and D is

(a) $5x^2 + 5y^2 - 8x - 16y - 36 = 0$

(b) $5x^2 + 5y^2 + 8x - 16y - 36 = 0$

(c) $5x^2 + 5y^2 + 8x + 16y - 36 = 0$

(d) $5x^2 + 5y^2 - 8x - 16y + 36 = 0$

Paragraph for Question No. 44 to 46

Consider the two circles $C_1 : x^2 + y^2 = r_1^2$ and $C_2 : x^2 + y^2 = r_2^2$ $(r_2 < r_1)$. Let $A(r_1, 0)$ be a fixed point on C_1 and 'B' be a variable point on C_2. The line BA meet the circle C_2 again at C. Then

44. Maximum value of BC^2 is

(a) $4r_1^2$ (b) $4r_2^2$

(c) $4r_2^2 - 4r_1^2$ (d) $r_1^2 - r_2^2$

45. Locus of midpoint of AB is

(a) $\left(x-\frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{4}$

(b) $\left(x-\frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{2}$

(c) $\left(x-\frac{r_2}{2}\right)^2 + y^2 = \frac{r_1^2}{2}$

(d) $\left(x-\frac{r_2}{2}\right)^2 + y^2 = \frac{r_1^2}{4}$

46. Locus of midpoint of AB when BC^2 is maximum is

(a) $x^2 + y^2 = (r_1 + r_2)^2$ (b) $x^2 + y^2 = (r_1 - r_2)^2$

(c) $x^2 + y^2 = \frac{(r_1 - r_2)^2}{4}$ (d) None of these

Paragraph for Question No. 47 to 49

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ.

47. The equation of the circle C is

(a) $(x-2\sqrt{3})^2 + (y-1)^2 = 1$

(b) $(x-2\sqrt{3})^2 + \left(y+\frac{1}{2}\right)^2 = 1$

(c) $(x-\sqrt{3})^2 + (y+1)^2 = 1$

(d) $(x-\sqrt{3})^2 + (y-1)^2 = 1$

48. The points E and F are

(a) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$

(b) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$

(c) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

(d) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

49. Equations of the sides QR and RP are

(a) $y = \frac{2}{\sqrt{x}}x + 1, y = -\frac{2}{\sqrt{3}}x - 1$

(b) $y = \frac{1}{\sqrt{3}}x, y = 0$

(c) $y = \frac{\sqrt{3}}{2}x + 1, y = -\frac{\sqrt{3}}{2}x - 1$

(d) $y = \sqrt{3}x, y = 0$

SECTION-IV

Matrix-Match Type

50. Match the following.

Column-I		Column-II	
(A)	If each of 4 circles touches the 3 lines $x + 2y - 1 = 0$, $2x - 3y + 5 = 0$ and $3x + y - 1 = 0$ of radii a, b, c, d and $a > b > c > d$, then $\frac{d}{a}+\frac{d}{b}+\frac{d}{c}=$	(p)	5
(B)	If $A(-2, 0)$ and $B(2, 0)$ are two points. Then the number of integral values of a if AB subtends obtuse angle at the point $(a, a + 1)$ is	(q)	2
(C)	If the lengths of the tangents from two points A & B to a circle are 3 & 4 and A & B are conjugate points, then $AB =$	(r)	4
(D)	A rectangle has dimensions 4 and 2 if two circles are constructed on these sides as diameters and d is the length of common chord, then $\sqrt{5}d =$	(s)	1

SECTION-V

Integer Answer Type

51. Tangents are drawn to the circle $x^2 + y^2 = 1$ at the points where it is met by the circles $x^2 + y^2 - (\lambda + 6)x + (8 - 2\lambda)y - 3 = 0$, λ being parameter. The locus of point of intersection of these tangents is $px + qy + 10 = 0$, then $p + q =$

52. 'C' is a circle that contains all the three circles $x^2 + y^2 + 12x + 4y + 31 = 0$, $x^2 + y^2 + 6x + 12y + 36 = 0$, $x^2 + y^2 - 4y - 5 = 0$ and having minimum radius. If 'D' is the centre of the circle 'C' and $E=\left(\frac{-13}{18},\frac{-11}{12}\right)$, then $ED^2 =$

53. The radius of the circle passing through the points (0, 0) and (2, 0) and touching the circle $x^2 + y^2 = 16$ is

54. Angle between the tangents drawn from a point P to the circle $x^2 + y^2 - 4x - 8y + 8 = 0$ is 60°, then the length of the tangent drawn from P to the circle is

55. If the line $3x - 4y - 8 = 0$ divides the circumference of the circle with centre $(2, -3)$ in the ratio 1 : 2, then the radius of the circle is

56. If r_1 and r_2 are the radii of smallest and largest circles which pass through (5, 6) and touches the circle $(x-2)^2 + y^2 = 4 \Rightarrow r_1 r_2 = \frac{41}{\lambda}$, then $\lambda =$

57. If two circles each of unit radius intersect orthogonally. The common area of the circles is $\frac{\pi}{\lambda}-1$. Then the value of $\lambda =$

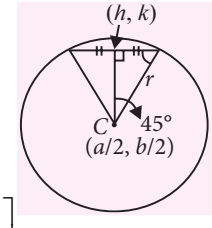
58. If the radius of circumcircle of the ΔTPQ where PQ is chord of contact corresponding to point T with respect to circle $x^2 + y^2 - 2x + 4y - 11 = 0$ is 6 units. Then the minimum distance of T from the director circle of the given circle is $12 - 4\sqrt{K} \Rightarrow K =$

59. For the circle $x^2 + y^2 = r^2$, the value of r for which the area enclosed by the tangents drawn from the point $P(6,8)$ to the circle and the chord of contact is maximum, is

SOLUTIONS

1. (c) : $r=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\frac{\sqrt{a^2+b^2}}{2}$

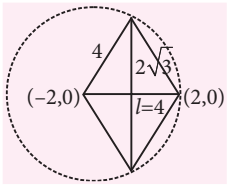
$$\sin 45^\circ = \frac{\sqrt{\left(h-\frac{a}{2}\right)^2+\left(k-\frac{b}{2}\right)^2}}{\frac{\sqrt{a^2+b^2}}{2}}$$



$$\Rightarrow \frac{1}{2} = 4\left[\frac{\frac{(2h-a)^2}{4}+\frac{(2k-b)^2}{4}}{a^2+b^2}\right]$$

simplify to get locus $x^2 + y^2 - ax - by + \frac{a^2+b^2}{8} = 0$

2. (a) : Circles with centre (2, 0) and (–2, 0) each with radius 4.

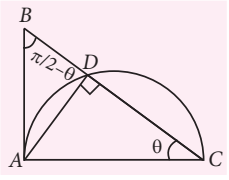


$\Rightarrow$ y-axis is their common chord.

The inscribed rhombus has its diagonals equal to 4 and $4\sqrt{3}$.

$\therefore\ A = \dfrac{d_1 d_2}{2} = 8\sqrt{3}$ sq. units

3. (d) : $\therefore\ \dfrac{AC}{AD} = \dfrac{1}{\sin\theta}$



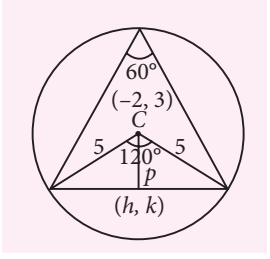
$\Rightarrow \dfrac{AD}{AB} = \sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\Rightarrow \dfrac{AD^2}{AC^2} + \dfrac{AD^2}{AB^2} = 1$

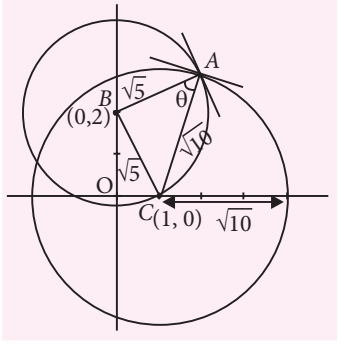
4. (b) : radius = 5 $\Rightarrow$ $p = 5\cos 60° = 2.5$

$\Rightarrow$ $(h + 2)^2 + (k - 3)^2 = 6.25$

$\therefore$ Locus is $(x + 2)^2 + (y - 3)^2 = 6.25$



5. (b) : $\cos\theta = \dfrac{10 + 5 - 5}{2\cdot\sqrt{10}\cdot\sqrt{5}} = \dfrac{1}{\sqrt{2}}$



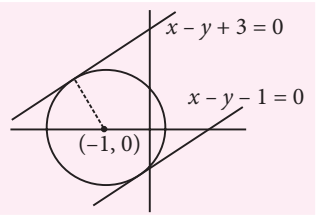
$\Rightarrow\ \theta = \dfrac{\pi}{4}$

6. (d) : $x^2 + y^2 + 2x - 1 = 0$; centre (–1, 0) and radius = $\sqrt{2}$

Line $x - y + c = 0$

$\Rightarrow \left|\dfrac{-1 + c}{\sqrt{2}}\right| = \sqrt{2} \Rightarrow |c - 1| = 2 \Rightarrow c - 1 = \pm 2$

$\Rightarrow$ $c = 3$ or -1



7. (c) : Equation of the two circles be

$(x - r)^2 + (y - r)^2 = r^2$

i.e. $x^2 + y^2 - 2rx - 2ry + r^2 = 0$, where $r = r_1$ and r_2.

or $x^2 + y^2 - 2r_1x - 2r_1y + r_1^2 = 0$

and $x^2 + y^2 - 2r_2x - 2r_2y + r_2^2 = 0$

Condition of orthogonality gives

$2r_1r_2 + 2r_1r_2 = r_1^2 + r_2^2 \Rightarrow 4r_1r_2 = r_1^2 + r_2^2$.

Circle passes through (a, b)

$\Rightarrow$ $a^2 + b^2 - 2ra - 2rb + r^2 = 0$

i.e. $r^2 - 2r(a + b) + a^2 + b^2 = 0$

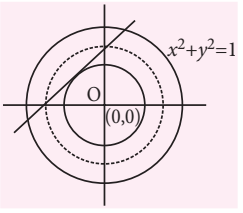
$r_1 + r_2 = 2(a + b)$ and $r_1r_2 = a^2 + b^2$

8. (d) : Equations of circle are

$x^2 + y^2 = 1$

$x^2 + y^2 = (1 - d)^2$

$x^2 + y^2 = (1 - 2d)^2$



$\Rightarrow$ solve any of circle with line $y = x + 1$

$\therefore\ x^2 + (1 + x)^2 = (1 - d)^2$

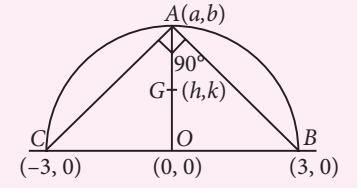
$\Rightarrow\ 2x^2 + 2x + 2d - d^2 = 0$

cuts at real and distinct points hence $\Delta > 0$

$\Rightarrow\ 2d^2 - 4d + 1 > 0 \Rightarrow\ d > \dfrac{2 + \sqrt{2}}{2}$ or $d < \dfrac{2 - \sqrt{2}}{2}$

9. (a) : Let $A = (a, b)$ and $G = (h, k)$. Now A, G, O are collinear.

$\Rightarrow\ h = \dfrac{2\cdot 0 + a}{3} \Rightarrow\ a = 3h$ and similarly $b = 3k$.

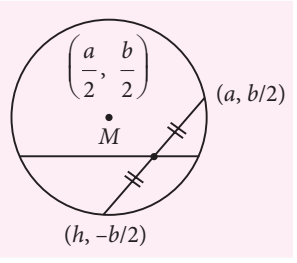


Now (a, b) lies on the circle $x^2 + y^2 = 9$

10. (c) : $(h, -b/2)$ must lie on the given circle

$\Rightarrow 2h^2 - 2ah + b^2 = 0$

For two distinct values of h, $D > 0 \Rightarrow a^2 > 2b^2$



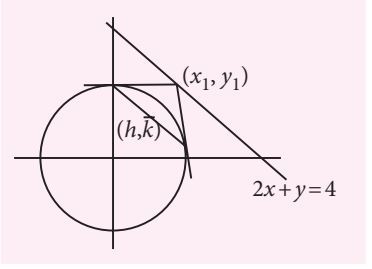
11. (c) : Let (x_1, y_1) lies on $2x + y = 4$

$\Rightarrow 2x_1 + y_1 = 4$ ....(1)

Chord of contact w.r.t. (x_1, y_1) is

$xx_1 + yy_1 = 1$

Also, equation of chord whose mid-point is (h, k)



$h^2 + k^2 = hx + ky$

$\therefore \quad \frac{x_1}{h} = \frac{y_1}{k} = \frac{1}{h^2+k^2} \Rightarrow x_1 = \frac{h}{h^2+k^2}; y_1 = \frac{k}{h^2+k^2}$

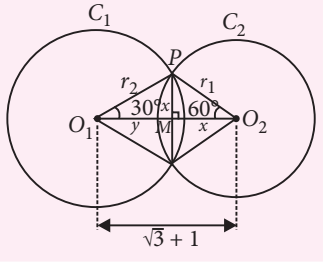
Substitute in (1), we get

$$2 \cdot \frac{h}{h^2+k^2} + \frac{k}{h^2+k^2} = 4$$

$\therefore$ locus is $4(x^2 + y^2) = 2x + y$

12. (c) : $\frac{y}{\sin 60°} = \frac{x}{\sin 30°} \Rightarrow y = \sqrt{3}x$

and $x(\sqrt{3}+1) = \sqrt{3}+1 \Rightarrow x = 1$



$\Rightarrow r_1 = x\sqrt{2} = \sqrt{2}$ and $r_2 = 2$

13. (d) : Here

$C_1 = (0, 0)$; $C_2 = (3\sqrt{3}, 3)$ and $r_1 = 2$, $r_2 = 4$

$\because \quad C_1C_2 = r_1 + r_2$

$\Rightarrow$ circle touch each other.

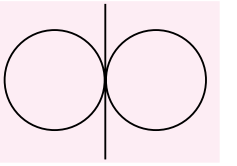
$\therefore$ Equation of transverse common tangent is,

$\sqrt{3}x + y - 4 = 0$

Comparing with

$x\cos\theta + y\sin\theta = 2$

$\Rightarrow \theta = \frac{\pi}{6}$



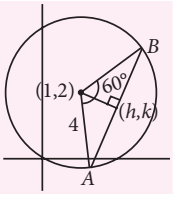
14. (c) : Let (h, k) be mid-point of chord.

$$\Rightarrow \frac{\sqrt{(1-h)^2+(2-k)^2}}{4} = \cos 30°$$

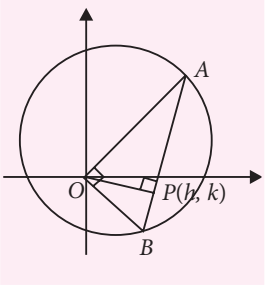
$$\Rightarrow (1-h)^2 + (2-k)^2 = \left(\frac{\sqrt{3}}{2} \times 4\right)^2$$

$\Rightarrow h^2 + 1 - 2h + k^2 + 4 - 4k = 12$

Hence locus is $x^2 + y^2 - 2x - 4y - 7 = 0$



15. (d) :



Let point be $P(h, k)$

Slope of perpendicular OP is k/h.

Slope of chord AB is $-(h/k)$

Equation of line AB is $(y-k) = \frac{-h}{k}(x-h)$

$\Rightarrow ky - k^2 + hx - h^2 = 0$

$$\Rightarrow \frac{ky+hx}{h^2+k^2} = 1 \qquad ...(1)$$

Homogenise the equation of circle by (1),

$$x^2 + y^2 - 4x\left(\frac{ky+hx}{h^2+k^2}\right) - 6y\left(\frac{ky+hx}{h^2+k^2}\right) - 3\left(\frac{ky+hx}{h^2+k^2}\right)^2 = 0$$

$\therefore$ It subtend a right angle at the origin.

$\therefore$ coeff. of x^2 + coeff. of $y^2 = 0$

$$\Rightarrow \left(1-\frac{4h}{h^2+k^2}-\frac{3h^2}{\left(h^2+k^2\right)^2}\right)+\left(1-\frac{6k}{h^2+k^2}-\frac{3k^2}{\left(h^2+k^2\right)^2}\right)=0$$

$$\Rightarrow 2-\frac{2(2h+3k)}{h^2+k^2}-\frac{3}{(h^2+k^2)}=0$$

$\Rightarrow 2(h^2+k^2)-4h-6k-3=0$

So locus is; $2x^2+2y^2-4x-6y-3=0$

16. (b) : We have, $OD = 1$, $OA = 2$

If $\angle AOD=\theta$ then $\cos\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$

$$\Rightarrow OA\cdot\sec\frac{\pi}{3}=2\cdot OA=4$$

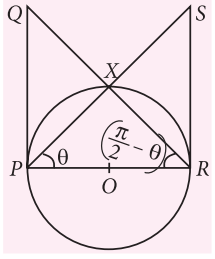
The locus of intersection of tangents is $x^2 + y^2 = 16$

17. (a) : In ΔPQR; $\tan\left(\frac{\pi}{2}-\theta\right)=\frac{PQ}{PR}$

$$\Rightarrow \cot\theta=\frac{PQ}{PR}$$

In ΔPSR; $\tan\theta=\frac{RS}{PR}$

Now, $\cot\theta\cdot\tan\theta=\left(\frac{PQ}{PR}\right)\left(\frac{RS}{PR}\right)$



$$\Rightarrow 1=\left(\frac{PQ}{PR}\right)\left(\frac{RS}{PR}\right)$$

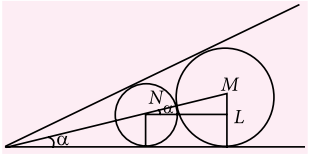
$\Rightarrow (PR)^2 = (PQ)(RS)$

$\Rightarrow 2r=\sqrt{PQ\cdot RS}$

18. (c) : From ΔMLN;

$$\sin\alpha=\frac{a-b}{a+b}$$

$$\Rightarrow\alpha=\sin^{-1}\left(\frac{a-b}{a+b}\right)$$



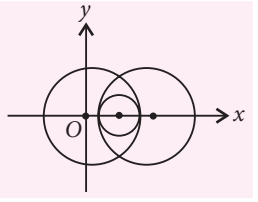
$\therefore$ Angle between direct common tangents is

$$\theta=2\alpha=2\sin^{-1}\left(\frac{a-b}{a+b}\right)$$

19. (b) : Maximum area is possible when the triangle PQR is an equilateral triangle of side $\sqrt{3}r$.

$\therefore$ Maximum area $=\frac{\sqrt{3}}{4}(3r^2)$

20. (d) : For the point P to enclose greatest area, the arbitrarily chosen point should be $\left(\frac{3}{2},0\right)$ and P should move in a circle of radius $\frac{1}{2}$. Locus of P is a circle of radius $\frac{1}{2}$.



$$\left(x-\frac{3}{2}\right)^2+(y-0)^2=\frac{1}{4}\Rightarrow x^2+y^2-3x+2=0$$

21. (b) : Length of chord of contact $=\frac{2RL}{\sqrt{R^2+L^2}}$

Equation of circle $x^2 + y^2 - 2x - 2y - 7 = 0$

Equation of chord of contact is

$4x + 4y - (x + 4) - (y + 4) - 7 = 0$

$\Rightarrow 3x + 3y - 15 = 0$

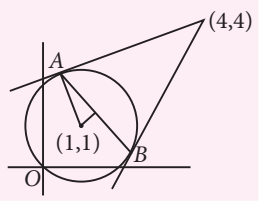
Radius = length of perpendicular from (1, 1)

$$=\left|\frac{6-15}{\sqrt{18}}\right|=\frac{9}{\sqrt{18}}=\frac{9}{3\sqrt{2}}=\frac{3}{\sqrt{2}}$$

$\therefore$ Length $\left(\frac{AB}{2}\right)$

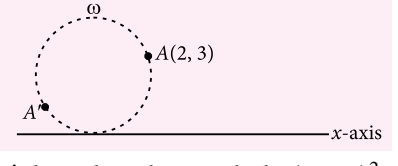
$$=\sqrt{r^2-\left(\frac{3}{\sqrt{2}}\right)^2}=\frac{3}{\sqrt{2}}$$

$\therefore AB=3\sqrt{2}$



22. (b) : Perimeter of ΔPLM = $PL + LC + CM + MP$
$= PL + LA + BM + MP = PA + PB$.

23. (b, c) : Let $A' = (x_1, y_1)$. The equation of ω is
$(x - x_1)(x - 2) + (y - y_1)(y - 3) = 0$.
$y = 0 \Rightarrow x^2 - (x_1 + 2)x + (2x_1 + 3y_1) = 0$
This equation has equal roots.
$\Rightarrow \Delta = 0 \Rightarrow (-(x_1 + 2))^2 = 4(2x_1 + 3y_1)$
$\Rightarrow (x_1 - 2)^2 = 12y_1$



Hence A' describes the parabola $(x - 2)^2 = 12y$

24. (a, b) : Line $|x - |y| - \alpha = 0$

$C(5, 0),\ r = \sqrt{25-21} = 2$

$d > r \Rightarrow |5-\alpha| > 2\sqrt{2} \Rightarrow 5-\alpha > 2\sqrt{2}$

or $5-\alpha < -2\sqrt{2}$

$\therefore\quad \alpha < 5-2\sqrt{2}$ or $5+2\sqrt{2} < \alpha$

25. (a, b) : The x-coordinates of the points of intersection of the line $y = mx + c$ and the circle $x^2 + y^2 = r^2$ are given by $x^2 + (mx + c)^2 = r^2$

or $(1 + m^2)x^2 + 2mcx + c^2 - r^2 = 0$ ....(i)

which, being quadratic in x, gives two values of x and hence two points of intersection.

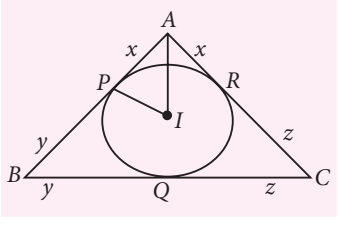
These points will be real and distinct if the discriminant of (i) is positive *i.e.*,

$4m^2c^2 - 4(1 + m^2)(c^2 - r^2) > 0 \Rightarrow c^2 < r^2(1 + m^2)$

$\Rightarrow\quad -r\sqrt{1+m^2} < c < r\sqrt{1+m^2}$

26. (a, b, c, d) : Let $AP = AR = x$,

$BP = BQ = y$ and $CQ = CR = z$



Then $x + y + z = \dfrac{a+b+c}{2} = 21$

Also, $y + z = 13$, $z + x = 14$, $x + y = 15$

$\Rightarrow\quad x = 8,\ y = 7,\ z = 6$

Also area of the triangle ABC is

$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21\times 8\times 7\times 6} = 84$

$\therefore$ The inradius is $r = \dfrac{\Delta}{s} = \dfrac{84}{21} = 4$

Area of the triangle APR is $\dfrac{r(s_{11})^{3/2}}{s_{11}+r^2}$

$= \dfrac{4\cdot(8)^3}{64+16} = \dfrac{128}{5}$

Area of the quadrilateral $APIR = r\sqrt{s_{11}} = 4 \times 8 = 32$

And $AI = \sqrt{AP^2 + PI^2} = \sqrt{64+16} = 4\sqrt{5}$

27. (a, c) : The given lines being parallel tangents to a circle, the diameter of the circle is equal to the distance between these lines, so the required radius is

$$\frac{1}{2}\times\left|\frac{4+\frac{7}{2}}{\sqrt{9+16}}\right| = \frac{1}{2}\times\frac{15}{2}\times\frac{1}{5} = \frac{3}{4}$$

The centre of the circle lies on the line parallel to the given lines at a distance of 3/4 from each of them. So let the equation be $3x - 4y + k = 0$ ...(i)

Then $\dfrac{k-4}{\sqrt{9+16}} = \pm\dfrac{3}{4} \Rightarrow k = 4 \pm \left(\dfrac{15}{4}\right) \Rightarrow k = \dfrac{1}{4}$ or $\dfrac{31}{4}$

For $k = \dfrac{1}{4}$, distance of (i) from the other line is $\dfrac{3}{4}$

Thus the centre lies on the line $12x - 16y + 1 = 0$

28. (a, b) : $x^2 + y^2 - 4y + 3 + \lambda x = 0$

$\Rightarrow$ common points are (0, 1), (0, 3)

$\therefore$ Equation of circle $S = 0$ is $x^2 + (y - 1)(y - 3) = 0$

i.e. $x^2 + y^2 - 4y + 3 = 0$

$\therefore$ Area of $S = 0$ is π and

radius of director circle $= \sqrt{2}(1) = \sqrt{2}$ units

29. (a, c) : Let tangents intersect at (α, β), then the line joining the two points at which tangents are drawn is

(i) common chord of C_1 and C_2 is $C_1 - C_2 = 0$

$\Rightarrow\quad 2x + 4y - 18 = 0.$

(ii) chord of contact when tangents are drawn from (α, β) to C_1

$\Rightarrow$ Equations $2x + 4y - 18 = 0$ and $\alpha x + \beta y - 25 = 0$ must be identical. On comparing, we get

$$\frac{\alpha}{2} = \frac{\beta}{4} = \frac{25}{18}$$

30. (a, c) : Equation of tangent is $\dfrac{x}{a} + \dfrac{y}{a} = 1$

$\Rightarrow\quad x + y = a$

$r = a \Rightarrow a = \pm 2\sqrt{2}$

31. (a, b) : $A(\alpha) = (a\cos\alpha, a\sin\alpha)$

$B(\beta) = (a\cos\beta, a\sin\beta)$

$C(\gamma) = (a\cos\gamma, a\sin\gamma)$

ΔABC is equilateral $S = G$

$$(0,0) = \left(\frac{a(\cos\alpha+\cos\beta+\cos\gamma)}{3}\right), \quad a\left(\frac{\sin\alpha+\sin\beta+\sin\gamma}{3}\right)$$

$\Rightarrow\quad \cos\alpha + \cos\beta + \cos\gamma = 0,$

$\sin\alpha + \sin\beta + \sin\gamma = 0$

32. (b, c, d) : Radius of S_1 and S_2 are roots of equation

$r^2 - 10r + 13 = 0$

$r_1 + r_2 = 10,\ r_1 r_2 = 13$

$(x-2)^2 + 4(y-3)^2 = 13$

Equation of S is $(x-2)^2 + (y-3)^2 = 13$

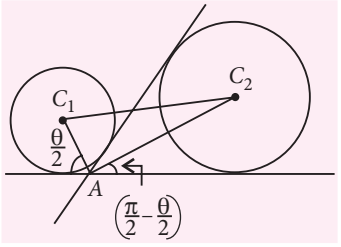
$\Rightarrow\ x^2 + y^2 - 4x - 6y = 0$

$\therefore\ \dfrac{r_1 + r_2}{2} = 5$. Also it cuts $y = x$

33. (a, b) : From the figure it is clear that $\angle C_1AC_2 = 90°$

Similarly $\angle C_1BC_2 = \angle C_1CC_2 = \angle C_1DC_2 = 90°$

Thus $ABCD$ is a cyclic quadrilateral with C_1C_2 as diameter $ABCD$ is clearly not a square.

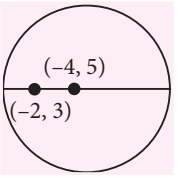


34. (a, c, d) : $x^2 + y^2 + 8x - 10y - 40 = 0$

$\therefore\ C(-4, 5),\ r = 9$

Also, on putting $(-2, 3)$ in the given circle, we get

$x^2 + y^2 + 8x - 10y - 40 < 0$



Distance of $(-4, 5)$ from the point $(-2, 3)$ is $2\sqrt{2}$

$\therefore\ a = 2\sqrt{2} + 9,\ b = -2\sqrt{2} + 9$

So, $a + b = 18,\ a - b = 4\sqrt{2},\ ab = 73$

35. (c) : Consider the circle in the first quadrant as $x^2 + y^2 - 2rx - 2ry + r^2 = 0$. It passes through (h, k), then $r^2 - 2r(h + k) + (h^2 + k^2) = 0$

Discriminant, $\Delta = 4(h + k)^2 - 4(h^2 + k^2) = 8hk \geq 0$

$\therefore$ The roots of r are real. But may be equal so 1 or 2 such circles will exist.

36. (c) : As like above, we get

$r^2 - 2r(a + b) + (a^2 + b^2) = 0$

We have $|r_1 - r_2| = C \ \Rightarrow (r_1 + r_2)^2 - 4r_1r_2 = C^2$

$\Rightarrow\ 8ab = C^2 \ \Rightarrow\ C = 2\sqrt{2ab}$

$\dfrac{a+b}{2} \geq \sqrt{ab} \Rightarrow a + b \geq \dfrac{C}{\sqrt{2}}$

37. (b) : The distance of the point (p, q) from the line $x \pm y = 0$ is $\dfrac{|p \pm q|}{\sqrt{2}}$

Hence the length of the common chord is

$$2\frac{|p \pm q|}{\sqrt{2}} = \sqrt{2}\,|p \pm q|$$

38. (c) **39. (c)** **40. (c)**

Let (x_1, y_1) be a point on C_2

Let (m, n) be the point which divides PQ in the ratio $p:q$, then

$$m = \frac{hq + x_1 p}{p+q} \text{ and } n = \frac{kq + y_1 p}{p+q}$$

$$x_1 = \frac{m(p+q) - hq}{p} \text{ and } y_1 = \frac{n(p+q) - kq}{p}$$

Since $Q(x_1, y_1)$ lies on $x^2 + y^2 = a^2$

we have $[m(p + q) - hq]^2 + [n(p + q) - kq]^2 = a^2p^2$

$$\Rightarrow \left[m - \frac{hq}{p+q}\right]^2 + \left[n - \frac{kq}{p+q}\right]^2 = \frac{a^2p^2}{(p+q)^2}$$

Locus of (m, n) is

$$\left[x - \frac{hq}{p+q}\right]^2 + \left[y - \frac{kq}{p+q}\right]^2 = \frac{a^2p^2}{(p+q)^2}$$

which is a circle C_3. If (α, β) denote the centre of C_3, then $\alpha = \dfrac{hq}{p+q}$ and $\beta = \dfrac{kq}{p+q}$. Since $P(h, k)$ lies on C_1,

$$\therefore\ q^2\left[\frac{p+q}{q}\right]^2(\alpha^2 + \beta^2) = p^2a^2$$

$\therefore$ Locus of (α, β) is $x^2 + y^2 = \dfrac{p^2a^2}{(p+q)^2}$

$$\frac{\text{radius of } C_1}{\text{radius of } C_4} = \sqrt{\frac{a^2p^2}{q^2} \times \frac{(p+q)^2}{a^2p^2}} = \frac{p+q}{q} = 1 + \frac{p}{q} = \frac{3}{2}$$

41. (a) : Lines $x + 2y + a = 0$ will intersect the circle $x^2 + y^2 = 4$ is

$$\left|\frac{0 + 0 + a}{\sqrt{1+4}}\right| < 2 \Rightarrow -2\sqrt{5} < a < 2\sqrt{5}$$

42. (d) : Let lines $x + 2y + a = 0$ and $12x - 6y - 41 = 0$ intersect at p, then $PA \cdot PB = PT^2$ and $PC \cdot PD = PT^{12}$ where T and T^1 are the points on the respective circles. A, B, C and D are concyclics.

$PA \cdot PB = PC \cdot PD \Rightarrow PT^2 = PT'^2$

Hence point P will lie on the radical axis of both the circles. Now equation of radical axis is $4x + 2y - 5 = 0$

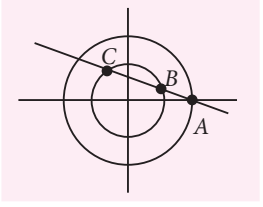
Since, radical axis and the lines $x + 2y + a = 0$ and $12x - 6y - 41 = 0$ are concurrent at P, we have

$$\begin{vmatrix} 4 & 2 & -5 \\ 1 & 2 & a \\ 12 & -6 & -41 \end{vmatrix} = 0 \Rightarrow a = 2$$

43. (a)

44. (b) : $(BC)^2 = (AC - AB)^2 = (AC + AB)^2 - 4AC \cdot AB$

$\therefore$ max. $(BC)^2 = 4r_2^2$, *i.e.*, BC becomes diameter of C_2.



45. (a) : Let B be the point (h, k), then midpoint of AB is given by

$$\left(\frac{h + r_1}{2}, \frac{k}{2}\right) = (x, y) \Rightarrow h = 2x - r_1, k = 2y$$

$$\therefore (2x - r_1)^2 + 4y^2 = r_2^2 \Rightarrow \left(x - \frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{4}$$

46. (d) : When BC is maximum, locus of midpoint of AB is a fixed point only.

47. (d) : Let $C(h, k)$ be the centre, then

$$\left|\frac{\sqrt{3}h + k - 6}{\sqrt{3+1}}\right| = 1 \Rightarrow \sqrt{3}h + k - 6 = -2$$

$$\Rightarrow \sqrt{3}h + k = 4$$

The point $(\sqrt{3}, 1)$ satisfies the equation.

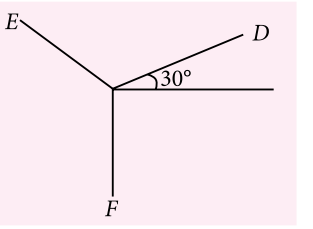
48. (a) : Slope of the line joining centre of circle to the point D is $\tan\theta = \dfrac{\frac{3}{2} - 1}{\frac{3\sqrt{3}}{2} - \sqrt{3}} = \dfrac{1}{\sqrt{3}}$

It makes an angle 30° with x-axis.

The points E and F will make angles 150° and 90° with x-axis.

Then E and F are

$$\frac{x - \sqrt{3}}{\cos 150°} = \frac{y - 1}{\sin 150°} = 1 \Rightarrow E = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$$



and $\dfrac{x - \sqrt{3}}{\cos(-90°)} = \dfrac{y - 1}{\sin(-90°)} = 1 \Rightarrow F = (\sqrt{3}, 0)$

49. (d) : The points E and F satisfy the equations given in option (d).

50. (A – s; B – q; C – p; D – r)

(A) Let $a = r_1$, $b = r_2$, $c = r_3$, $d = r$

$$\frac{d}{a} + \frac{d}{b} + \frac{d}{c} = r\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = r\left(\frac{1}{r}\right) = 1$$

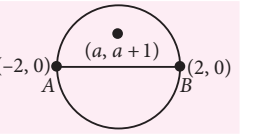
(B) The point $(a , a + 1)$ lies inside the circle $x^2 + y^2 = 4$

$\Rightarrow S_{11} < 0$

$\Rightarrow a^2 + (a + 1)^2 - 4 < 0$

$\Rightarrow 2a^2 + 2a - 3 < 0$

$$\Rightarrow a \in \left(\frac{-1 - \sqrt{7}}{2}, \frac{-1 + \sqrt{7}}{2}\right)$$



(C) If l_1, l_2 are lengths of tangents, then

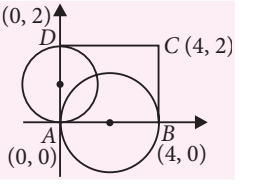
$AB^2 = l_1^2 + l_2^2$

$\Rightarrow AB^2 = 9 + 16 \Rightarrow AB = 5$

(D) Common chord

$$= \frac{2r_1r_2}{C_1C_2} = \frac{2 \cdot 2 \cdot 1}{\sqrt{5}}$$

$$\Rightarrow d = \frac{4}{\sqrt{5}}$$



51. (1) : $xx_1 + yy_1 - 1 = 0$ ...(1)

$S - S' = 0$

$\Rightarrow (\lambda + 6)x + (2\lambda - 8)y + 2 = 0$ ...(2)

(1) and (2) represent the same line.

$\therefore -2x_1 = \lambda + 6, -2y_1 = 2\lambda - 8$

$\therefore -2x_1 - 6 = -y_1 + 4$

$\Rightarrow 2x - y + 10 = 0$

$$\therefore \left.\begin{matrix} p = 2 \\ q = -1 \end{matrix}\right\} \Rightarrow p + q = 1$$

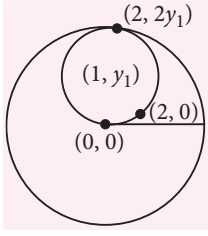
52. (2) : $D=\left(\frac{-31}{18},\frac{-23}{12}\right)$, $E=\left(\frac{-13}{18},\frac{-11}{12}\right)$

$\Rightarrow DE^2 = 2$

53. (2) : Centre of the required circle be $(1, y_1)$

$\Rightarrow$ $(2, 2y_1)$ lies in $x^2 + y^2 = 16$ $\Rightarrow$ $y_1^2 = 3$

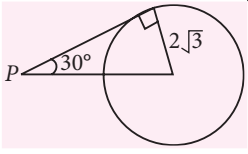
$\Rightarrow$ radius $=\sqrt{1+3}=2$



54. (6) : $r=\sqrt{4+16-8}=2\sqrt{3}$

$\therefore$ Required length of tangent $=\frac{r}{\tan 30^\circ}$

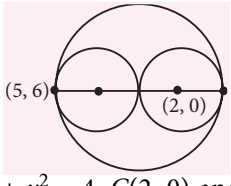
$=2\sqrt{3}\times\sqrt{3}=6$



55. (4) : $d=\frac{|6+12-8|}{\sqrt{3^2+4^2}}=2$

$2\theta=\frac{1}{3}(360^\circ)=120^\circ$ $\therefore$ $\cos 60^\circ=\frac{d}{r}\Rightarrow r=4$

56. (4) :



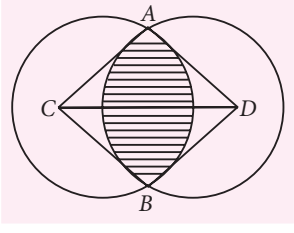
$(x - 2)^2 + y^2 = 4$, $C(2, 0)$ and $r = 2$

Distance between (2,0) and (5, 6) is

$\sqrt{9+36}=\sqrt{45}=3\sqrt{5}$

$\therefore$ $r_1r_2=\frac{3\sqrt{5}-2}{2}\times\frac{3\sqrt{5}+2}{2}=\frac{41}{4}$

57. (2) :



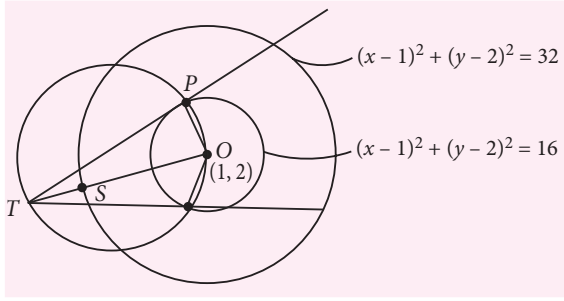
$ABCD$ is square

$\therefore$ Required area = Area of sector ABC + Area of sector BDA – Area of square $ABCD$

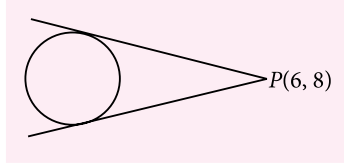
$=2\left(\frac{\pi}{4}\right)-1=\frac{\pi}{2}-1$

58. (2) : $OS=4\sqrt{2}$

Required distance $TS = OT - SO = 12-4\sqrt{2}$



59. (5) : Area of the triangle $=\frac{r\cdot(x_1^2+y_1^2-r^2)^{3/2}}{x_1^2+y_1^2}$



$A=\frac{r.(100-r^2)^{3/2}}{100}$

$\Rightarrow$ $\frac{dA}{dr}=\frac{r.\frac{3}{2}(100-r^2)^{1/2}(-2r)+(100-r^2)^{3/2}}{100}$

$=\frac{(100-r^2)^{1/2}}{100}(-3r^2+100-r^2)=0 \Rightarrow r=5$

$\therefore$ Area is maximum, when $r = 5$

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Class XII

Vectors and 3D Geometry

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

VECTORS

DEFINITIONS

- A vector may be described as a quantity having both magnitude and direction. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the initial point and B is called the terminal point. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.
- **Zero vector :** A vector of zero magnitude *i.e.* which has the same initial and terminal point, is called a zero vector. It is denoted by $\vec{O}$.
- **Unit vector :** A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector and is denoted by $\hat{a}$. Symbolically $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$.
- **Equal vectors :** Two vectors are said to be equal, if they have the same magnitude, direction and represent the same physical quantity.
- **Collinear vectors :** Two vectors are said to be collinear, if their directed line segments are parallel disregards to their direction. Collinear vectors are also called **parallel vectors**. If they have the same direction they are named as like vectors otherwise unlike vectors.

  Symbolically, two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a}=K\vec{b}$, where $K \in R$.
- **Coplanar vectors :** A given number of vectors are called coplanar if their all line segments are parallel to the same plane.

  Note that "Two vectors are always coplanar".
- **Position vector :** Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B respectively, then

  $\overrightarrow{AB}=\vec{b}-\vec{a}$ = P.V. of B – P.V. of A

VECTOR ADDITION

- If two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are represented by $\vec{a}$ and $\vec{b}$, then their sum $\overrightarrow{OC}$ is a vector represented by $\vec{a}+\vec{b}$ where OC is the diagonal of the parallelogram $OACB$.
- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative)
- $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associative)
- $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$
- $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$

MULTIPLICATION OF VECTOR BY SCALARS

If $\vec{a}$ is a vector and m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times to that of $\vec{a}$. This multiplication is called scalar multiplication. If $\vec{a}$ and $\vec{b}$ are vectors and m, n are scalars, then

$$m(\vec{a})=(\vec{a})m=m\vec{a}$$

$$m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$$

$$(m+n)\vec{a}=m\vec{a}+n\vec{a}$$

$$m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$$

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SECTION FORMULA

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector of a point which divides AB in the ratio $m:n$ is given by $\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}$.

Note : Position vector of mid-point of $AB = \dfrac{\vec{a}+\vec{b}}{2}$.

DIRECTION COSINES

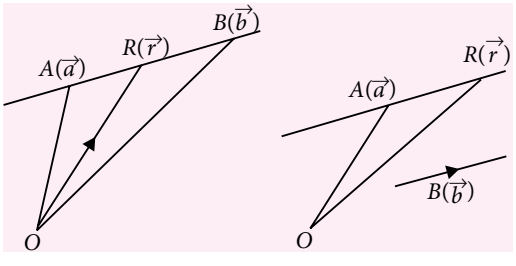
Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and α, β, γ are the angles which this vector makes with the +ve directions of OX, OY and OZ called direction angles, then their cosines are called the direction cosines.

i.e., $\cos\alpha = \dfrac{a_1}{|\vec{a}|}$, $\cos\beta = \dfrac{a_2}{|\vec{a}|}$, $\cos\gamma = \dfrac{a_3}{|\vec{a}|}$

Note that, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

VECTOR EQUATION OF A LINE

- Parametric vector equation of a line passing through two points $A(\vec{a})$ and $B(\vec{b})$ is given by, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, where t is a parameter.
- If the line passes through the point $A(\vec{a})$ and is parallel to the vector $\vec{b}$, then its equation is, $\vec{r} = \vec{a} + t\vec{b}$.
- Note that the equations of the bisectors of the angle between the lines,
  $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ is
  $\vec{r} = \vec{a} + t(\vec{b} + \vec{c})$ and $\vec{r} = \vec{a} + p(\vec{c} - \vec{b})$



TEST OF COLLINEARITY

Three points A, B, C with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively are collinear, if and only if their exist scalars x, y, z not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where $x + y + z = 0$.

SCALAR PRODUCT OF TWO VECTORS

- $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ $(0 \le \theta \le \pi)$,
  Note that if θ is acute, then $\vec{a}\cdot\vec{b} > 0$ and if θ is obtuse, then $\vec{a}\cdot\vec{b} < 0$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)
- $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive)
- $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \neq 0, \vec{b} \neq 0)$
- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$; $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

REMARKS

Vector component of $\vec{a}$ along $\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ and perpendicular to $\vec{b} = \vec{a} - \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$.

- The angle ϕ between $\vec{a}$ and $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$, $0 \le \phi \le \pi$.
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then
  $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
  $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$,
  $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$.

REMARKS

- Maximum value of $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$.
- Minimum value of $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$.
- Any vector $\vec{a}$ can be written as ,
  $\vec{a} = (\vec{a}\cdot\hat{i})\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\hat{k}$.
- A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is
  $\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}$.
- Hence, bisector of the interior angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\lambda(\hat{a} + \hat{b})$, where $\lambda \in R^+$.
  Bisector of the exterior angle between two vectors $\vec{a}$ and $\vec{b}$ is $\lambda(\hat{a} - \hat{b})$, $\lambda \in R^+$.

VECTOR PRODUCT OF TWO VECTORS

- If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them, then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}$ and $\hat{n}$ forms a right handed screw system .
- **Lagrange's identity:** For any two vectors $\vec{a}$ and $\vec{b}$;
$$(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = \begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$$
- **Formulation of vector product in terms of scalar product**
The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that
(i) $|\vec{c}| = \sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$
(ii) $\vec{c}\cdot\vec{a} = 0, \vec{c}\cdot\vec{b} = 0$ and
(iii) $\vec{a}, \vec{b}, \vec{c}$ form a right handed system
- $\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a}\neq 0,\ \vec{b}\neq 0)$ *i.e.* $\vec{a} = K\vec{b}$, where K is a scalar.
- $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ (not commutative)
- $(m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b}) = m(\vec{a}\times\vec{b})$, where m is a scalar.
- $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ (distributive)
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$
- $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$
- Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.
- Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\hat{n} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$
- A vector of magnitude r and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$
- If θ is the angle between $\vec{a}$ and $\vec{b}$, then
$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

VECTOR AREA

- If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of 3 points A, B and C, then the vector area of triangle ABC
$$= \frac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$$
The points A, B and C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 0$
- Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$.

SHORTEST DISTANCE BETWEEN TWO LINES

If two lines directed along $\vec{p}$ and $\vec{q}$ in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called skew lines. For Skew lines, the direction of the shortest distance would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\overrightarrow{AB}$ along the direction of the line of shortest distance, $\overrightarrow{LM}$ is parallel to $\vec{p}\times\vec{q}$.

i.e. $\overrightarrow{LM} = |\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{LM}|$

$$= |\text{Projection of } \overrightarrow{AB} \text{ on } \vec{p}\times\vec{q}|$$

$$= \left|\frac{\overrightarrow{AB}\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right| = \left|\frac{(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

- The two lines will intersect only if shortest distance = 0 *i.e.* $(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q}) = 0$ *i.e.* $\vec{b}-\vec{a}$ lies in the plane containing $\vec{p}$ and $\vec{q}$
$\Rightarrow [(\vec{b}-\vec{a})\ \vec{p}\ \vec{q}] = 0$
- If two lines are given by $\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b}$ *i.e.* they are parallel then,
$$d = \left|\frac{\vec{b}\times(\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT

- The scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as

$(\vec{a}\times\vec{b})\cdot\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and ϕ is the angle between $(\vec{a}\times\vec{b})$ and $\vec{c}$. It is also defined as $[\vec{a}\,\vec{b}\,\vec{c}]$, spelled as box product.

- Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}$, $\vec{b}$ and $\vec{c}$ *i.e.* $V = [\vec{a}\ \vec{b}\ \vec{c}]$
- In a scalar triple product, the position of dot and cross can be interchanged
  i.e. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$
  or $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$
- $\vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$ *i.e.* $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

  In general, if
  $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$
  and $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$, then
  $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\,\vec{m}\,\vec{n}]$; where $\vec{l}$, $\vec{m}$, $\vec{n}$ are non-coplanar vectors.
- If $\vec{a}, b, \vec{c}$ are coplanar $\Leftrightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.
- Scalar product of three vectors, two of which are equal or parallel is 0 *i.e.* $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

REMARKS

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

- $[\hat{i}\,\hat{j}\,\hat{k}] = 1$
- $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$
- $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$
- The volume of the tetrahedron $OABC$ with O as origin and the position vectors of A, B and C being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$
- The position vector of the centroid of a tetrahedron if the position vectors of its angular vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are given by $\frac{1}{4}[\vec{a} + \vec{b} + \vec{c} + \vec{d}]$.

  Note : This is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case, the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron .

  Remember that : $[\vec{a}-\vec{b}\quad \vec{b}-\vec{c}\quad \vec{c}-\vec{a}] = 0$ and $[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

VECTOR TRIPLE PRODUCT

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector and is called a vector triple product .

Geometrical Interpretation of $\vec{a}\times(\vec{b}\times\vec{c})$

Consider the expression $\vec{a}\times(\vec{b}\times\vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ and $(\vec{b}\times\vec{c})$. Now $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ and $(\vec{b}\times\vec{c})$ but $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$, therefore $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector lies in the plane of $\vec{b}$ and $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a}\times(\vec{b}\times\vec{c})$ in terms of $\vec{b}$ and $\vec{c}$.

i.e. $\vec{a}\times(\vec{b}\times\vec{c}) = x\vec{b} + y\vec{c}$, where x and y are scalars.

- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$
- $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$
- $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

LINEAR COMBINATIONS

Linearly independence and dependence of vectors :

- Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$, then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots$ for any $x, y, z, \ldots \in R$. We have the following results :
- **Fundamental theorem in plane :** Let $\vec{a}, \vec{b}$ be non-zero, non-collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$

i.e. there exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} = \vec{r}$.

- **Fundamental theorem in space :** Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ *i.e.* there exist some unique $x,y,z \in R$ such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.
- If $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are n non-zero vectors, and $k_1, k_2, ..., k_n$ are n scalars and if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_n\vec{x}_n = 0$
$\Rightarrow k_1 = 0, k_2 = 0, ..., k_n = 0$
then we say that vectors $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are linearly independent vectors.
- If $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are not linearly independent, then they are said to be linearly dependent vectors. *i.e.* if $k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_n\vec{x}_n = 0$ and if there exists at least one $k_r \neq 0$, then $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are said to be linearly dependent.

REMARKS

- If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$, then $\vec{a}$ is expressed as a linear combination of vectors $\hat{i}, \hat{j}, \hat{k}$. Also, $\vec{a}, \hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors is a linearly dependent system.
- $\hat{i}, \hat{j}, \hat{k}$ are linearly independent set of vectors. For $K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = 0 \Rightarrow K_1 = 0 = K_2 = K_3$.
- Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent $\Rightarrow \vec{a}$ is parallel to $\vec{b}$ *i.e.* $\vec{a} \times \vec{b} = 0 \Rightarrow$ linear dependence of $\vec{a}$ and $\vec{b}$. Conversely if $\vec{a} \times \vec{b} \neq 0$ then $\vec{a}$ and $\vec{b}$ are linearly independent.
- If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar *i.e.* $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. Conversely, if $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$, then the vectors are linearly independent.

COPLANARITY OF VECTORS

- Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ where, $x + y + z + w = 0$.

RECIPROCAL SYSTEM OF VECTORS

- $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non-coplanar vectors such that $\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$, then the two systems are called reciprocal system of vectors.

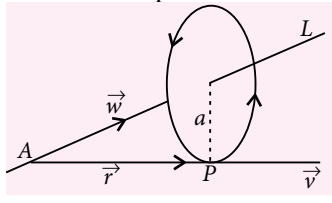
Note : $\vec{a}' = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{b}' = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{c}' = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

EQUATION OF A PLANE

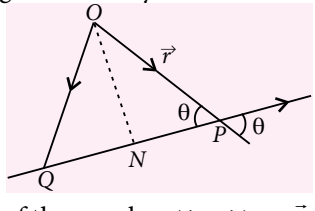
- The equation $(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$ represents a plane containing the point with position vector $\vec{r}_0$, where $\vec{n}$ is a vector normal to the plane.
$\vec{r} \cdot \vec{n} = d$ is the general equation of a plane.
- Angle between the 2 planes is the angle between 2 normals drawn to the planes and the angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

APPLICATION OF VECTORS

- Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as $\overrightarrow{W} = \vec{F} \cdot \vec{s}$.
- The tangential velocity $\vec{v}$ of a body moving in a circle is given by $\vec{v} = \vec{w} \times \vec{r}$, where $\vec{r}$ is the position vector of the point P.



- The moment of $\vec{F}$ about 'O' is defined as $\vec{M} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector of P w.r.t. 'O'. The direction of $\vec{M}$ is along the normal to the plane OPN such that $\vec{r}$, $\vec{F}$ and $\vec{M}$ form a right handed system.



- Moment of the couple $= (\vec{r}_1 - \vec{r}_2) \times \vec{F}$, where $\vec{r}_1$ and $\vec{r}_2$ are position vectors of the point of the application of the forces $\vec{F}$ and $-\vec{F}$.

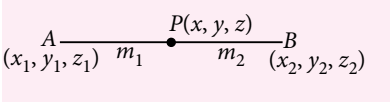
3-D COORDINATE GEOMETRY

- **General**

  Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

  $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Section formula**



$$x = \frac{m_2 x_1 + m_1 x_2}{m_1 + m_2}, \quad y = \frac{m_2 y_1 + m_1 y_2}{m_1 + m_2}$$

$$z = \frac{m_2 z_1 + m_1 z_2}{m_1 + m_2}$$

(For external division take –ve sign)

- **Direction cosines and direction ratios of a line**

  (i) Direction cosine of a line has the same meaning as d.c's of a vector.

  (ii) Any three numbers a, b, c proportional to the direction cosines are called the direction ratios *i.e.*

  $$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

  same sign either +ve or –ve should be taken through out.

  Note that d.r's of a line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$.

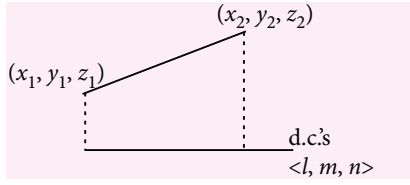
  (iii) If θ is the angle between the two lines whose d.c's are $<l_1, m_1, n_1>$ and $<l_2, m_2, n_2>$, then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

  Hence if lines are perpendicular, then $l_1l_2 + m_1m_2 + n_1n_2 = 0$

  If lines are parallel, then $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

  Note that if three lines are coplanar,

  then $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$



- Projection of the join of two points (x_1, y_1, z_1) and (x_2, y_2, z_2) on a line with d.c's $<l, m, n>$ are $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

PLANE

- General equation of degree one in x, y, z, *i.e.* $ax + by + cz + d = 0$ represents a plane.
- Equation of a plane passing through (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.
- Equation of a plane if its intercepts on the coordinate axes are x_1, y_1, z_1 is $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$.
- Equation of a plane if the length of the perpendicular from the origin on the plane is p and d.c's of the perpendicular as $<l, m, n>$ is $lx + my + nz = p$
- **Parallel and perpendicular planes :** Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

  (i) perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

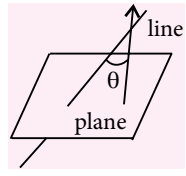
  (ii) parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

  (iii) coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$

- Angle between a plane and a line is the complement of the angle between the normal to the plane and the line. If line : $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane : $\vec{r} \cdot \vec{n} = d$, then

  $$\cos(90° - \theta) = \sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}$$

  where θ is the angle between the line and normal to the plane.



- Length of the perpendicular from a point (x_1, y_1, z_1) to a plane $ax + by + cz + d = 0$ is

$$p = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

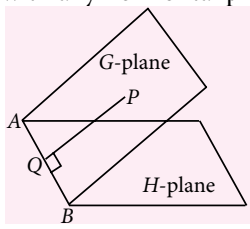
- Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is
$$\left| \frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}} \right|$$
- Planes bisecting the angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by
$$\left| \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} \right| = \pm \left| \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$
Of these two bisecting planes, one bisects the acute and the other obtuse angle between the given planes.
- Equation of a plane through the intersection of two planes P_1 and P_2 is given by $P_1 + \lambda P_2 = 0$

STRAIGHT LINE IN SPACE

- Equation of a line through $A(x_1, y_1, z_1)$ and having direction cosines $<l, m, n>$ are
$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$
and the equation of a line through (x_1, y_1, z_1) and (x_2, y_2, z_2) is
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$
- Intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ together represent the unsymmetrical form of the straight line.
- General equation of the plane containing the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is
$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$,
where $Al + Bm + Cn = 0$.

LINE OF GREATEST SLOPE

- AB is the line of intersection of plane G and H is the horizontal plane. Line of greatest slope on a given plane, drawn through a given point on the plane, is the line through the point 'P' perpendicular to the line of intersection of the given plane with any horizontal plane.



PROBLEMS
SECTION-I
Single Correct Answer Type

1. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the three vectors having magnitudes 1, 5 and 3 respectively such that the angle between $\vec{a}$ and $\vec{b}$ is θ and $\vec{a} \times (\vec{a} \times \vec{b}) = \vec{c}$, then $\tan\theta$ is equal to
(a) 0 (b) 2/3 (c) 3/5 (d) 3/4

2. The perpendicular distance of a corner of a unit cube from a diagonal not passing through it is
(a) $\sqrt{\frac{3}{2}}$ (b) $\sqrt{\frac{2}{3}}$ (c) $\sqrt{\frac{3}{4}}$ (d) $\sqrt{\frac{4}{3}}$

3. If $\vec{p}, \vec{q}$ are two non-collinear vectors such that $(b-c)(\vec{p} \times \vec{q}) + (c-a)\vec{p} + (a-b)\vec{q} = \vec{0}$ where a, b, c are lengths of sides of a triangle, then the triangle is
(a) right angled (b) obtuse angled
(c) equilateral (d) right angled isosceles

4. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{d} = \hat{i} + \hat{j} - \hat{k}$. Then, the line of intersection of planes one determined by $\vec{a}, \vec{b}$ and other determined by $\vec{c}, \vec{d}$ is perpendicular to
(a) x-axis (b) y-axis
(c) both x and y-axes (d) both y and z-axes

5. A parallelopiped is formed by planes drawn parallel to coordinate axes through the points A = (1, 2, 3) and B = (9, 8, 5). The volume of parallelopiped is equal to (in cubic units)
(a) 192 (b) 48 (c) 32 (d) 96

6. A line passing through $A(1, 2, 3)$ and having direction ratios (3, 4, 5) meets a plane $x + 2y - 3z = 5$ at B, then length of AB is equal to
(a) $\frac{9}{4}$ (b) $\frac{11}{4\sqrt{2}}$ (c) $\frac{13}{4}$ (d) $\frac{45\sqrt{2}}{4}$

7. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors and $\vec{d}$ be a non-zero vector, which is perpendicular to $\vec{a}+\vec{b}+\vec{c}$. Now, if $\vec{d}=(\sin x)(\vec{a}\times\vec{b})+(\cos y)(\vec{b}\times\vec{c})+2(\vec{c}\times\vec{a})$, then minimum value of x^2+y^2 is equal to

(a) π^2 (b) $\frac{\pi^2}{2}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{5\pi^2}{4}$

8. If the four faces of a tetrahedron are represented by the equations $\vec{r}\cdot(\alpha\hat{i}+\beta\hat{j})=0$, $\vec{r}\cdot(\beta\hat{j}+\gamma\hat{k})=0$, $\vec{r}\cdot(\gamma\hat{k}+\alpha\hat{i})=0$ and $\vec{r}\cdot(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})=p$, then volume of the tetrahedron (in cubic units) is

(a) $\left|\frac{p^3}{6\alpha\beta\gamma}\right|$ (b) $\left|\frac{4p^3}{6\alpha\beta\gamma}\right|$

(c) $\left|\frac{3p^3}{6\alpha\beta\gamma}\right|$ (d) none of these

9. The equation of a plane that passes through (1, 2, 3) and is at maximum distance from (–1, 1, 1) is

(a) $2x - y + 2z = 6$ (b) $2x - y + z = 3$

(c) $2x + y + 2z = 10$ (d) none of these

10. The distance of the point $P(3, 8, 2)$ from the line $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{3}$ measured parallel to the plane $3x + 2y - 2z + 15 = 0$ is

(a) 7 (b) 2 (c) 9 (d) 5

11. The locus of a point which moves in such a way that its distance from the line $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ is twice the distance from the plane $x + y + z = 0$ is

(a) $x^2 + y^2 + z^2 - 5x - 3y - 3z = 0$

(b) $x^2 + y^2 + z^2 + 5x + 3y + 3z = 0$

(c) $x^2 + y^2 - z^2 - 5xy - 3yz - 3zx = 0$

(d) $x^2 + y^2 - z^2 + 5xy + 3yz + 3zx = 0$

12. A variable plane intercepts on the axes, the sum of whose squares is 9. Then locus of the foot of the perpendicular from origin to plane is $(x^{-2}+y^{-2}+z^{-2})(x^2+y^2+z^2)^2 = K$. Then K is

(a) 3 (b) 9 (c) 27 (d) 81

13. A triangle ABC has vertices $A(1, -2, 2)$, $B(1, 4, 0)$ and $C(-4, 1, 1)$. The vector BM, where M is the foot of the altitude drawn from B to AC is

(a) $-\frac{20}{3}\hat{i}-10\hat{j}+\frac{10}{3}\hat{k}$ (b) $-\frac{10}{7}\hat{i}-\frac{30}{7}\hat{j}+\frac{10}{7}\hat{k}$

(c) $\frac{20}{7}\hat{i}+5\hat{j}-\frac{10}{7}\hat{k}$ (d) $-\frac{20}{7}\hat{i}-\frac{30}{7}\hat{j}+\frac{10}{7}\hat{k}$

14. Angle between the lines $3x + 2y + z - 5 = 0 = x + y - 2z - 3$ and $x - y + 4z = 0 = x + y - 4z$ is equal to

(a) $\cos^{-1}\left(\frac{51}{5\sqrt{29}}\right)$ (b) $\cos^{-1}\left(\frac{29}{5\sqrt{51}}\right)$

(c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\frac{\pi}{2}$

15. Let $A(0, 0, 1)$, $B(0, 1, 0)$ and $C(1, 1, 1)$ are the points in a plane. Then the equation of the plane perpendicular to the plane ABC and passing through A and B is

(a) $-x + y + z + 1 = 0$ (b) $x - y - z - 1 = 0$

(c) $-x + y + z - 1 = 0$ (d) $2x + y + z - 1 = 0$

16. A ray of light is sent through the point $P(1, 2, 3)$ and is reflected on the XY plane. If the reflected ray passes through the point $Q(3, 2, 5)$, then the equation of the reflected ray is

(a) $\frac{x-3}{1}=\frac{y-2}{0}=\frac{z-5}{1}$ (b) $\frac{x-3}{1}=\frac{y-2}{0}=\frac{z-5}{-4}$

(c) $\frac{x-3}{1}=\frac{y-2}{0}=\frac{z-5}{4}$ (d) $\frac{x-1}{1}=\frac{y-2}{0}=\frac{z-3}{4}$

17. The intercepts made by a plane, P, on the co-ordinate axes are a, b, c. The co-ordinate axes are rotated about the origin by an angle θ. The plane P now makes the intercepts a', b', c' on the (new) coordinate axes. Which of the following statements is true?

(a) $a + b + c = a' + b' + c'$

(b) $abc = a'\,b'\,c'$

(c) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$

(d) $bc + ca + ab = b'c' + c'\,a' + a'\,b'$.

18. The distance of the point (3, 4, 5) from the plane $x + y + z = 2$ measured parallel to the line $2x = y = z$ is

(a) $4\sqrt{3}$ (b) 6 (c) $5\sqrt{2}$ (d) 12

19. For the line $l: \frac{x-1}{3}=\frac{y+1}{2}=\frac{z-3}{-1}$ and the plane $P: x - 2y - z = 0$; which of the following statements is true?

(a) l lies in P (b) l is parallel to P

(c) l is perpendicular to P (d) none of these

20. If P, Q, R are the images of point $A(a, b, c)$ in XY, YZ and ZX planes respectively and 'G' is the centroid of ΔPQR, then area of ΔAOG = (O refers to origin)

(a) 0 (b) $a^2 + b^2 + c^2$

(c) $\frac{2}{3}(a^2+b^2+c^2)$ (d) $\frac{3}{4}(a^2+b^2+c^2)$

21. The equation of plane through points (1, 0, –1), (3, 2, 2) and parallel to the line $\frac{x-1}{1}=\frac{1-y}{2}=\frac{z-2}{3}$ is

(a) $4x - y - 2z - 6 = 0$ (b) $5x + y - 2z - 7 = 0$

(c) $4x + y - 2z - 7 = 0$ (d) $4x - y + 2z + 1 = 0$

SECTION-II

More than One Correct Answer Type

22. If $\alpha(\vec{a}\times\vec{b})+\beta(\vec{b}\times\vec{c})+\gamma(\vec{c}\times\vec{a})=0$, then

(a) $\vec{a},\vec{b},\vec{c}$ are coplanar if all of $\alpha, \beta, \gamma \neq 0$

(b) $\vec{a},\vec{b},\vec{c}$ are coplanar if any one of $\alpha, \beta, \gamma \neq 0$

(c) $\vec{a},\vec{b},\vec{c}$ are non-coplanar for any α, β, γ

(d) none of these

23. If $\vec{a},\vec{b},\vec{c}$ are three non-zero vectors, then which of the following statement(s) is/are true ?

(a) $\vec{a}\times(\vec{b}\times\vec{c}), \vec{b}\times(\vec{c}\times\vec{a}), \vec{c}\times(\vec{a}\times\vec{b})$ form a right handed system.

(b) $\vec{c}, (\vec{a}\times\vec{b})\times\vec{c}, \vec{a}\times\vec{b}$ form a right handed system.

(c) $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}<0$, if $\vec{a}+\vec{b}+\vec{c}=\vec{0}$.

(d) $\frac{(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})}{(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{c})}=-1$, if $\vec{a}+\vec{b}+\vec{c}=\vec{0}$.

24. Let $\vec{a}$ and $\vec{b}$ be two given perpendicular vectors, which are non-zero. A vector $\vec{r}$ satisfying the equation $\vec{r}\times\vec{b}=\vec{a}$ can be

(a) $\vec{b}-\frac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$ (b) $2\vec{b}-\frac{(\vec{a}\times\vec{b})}{|\vec{b}|^2}$

(c) $|\vec{a}|\vec{b}-\frac{(\vec{a}\times\vec{b})}{|\vec{b}|^2}$ (d) $|\vec{b}|\vec{b}-\frac{(\vec{a}\times\vec{b})}{|\vec{b}|^2}$

25. If $\vec{a}$ and $\vec{b}$ are any two unit vectors, then possible integer(s) in the range of

$\frac{3|\vec{a}+\vec{b}|}{2}+2|\vec{a}-\vec{b}|$ is

(a) 2 (b) 3 (c) 4 (d) 5

26. A non-zero vector $\vec{a}$ is parallel to the line of intersection of the plane determined by the vectors $\hat{i}, \hat{i}+\hat{j}$ and the plane determined by the vectors $\hat{i}-\hat{j}, \hat{i}+\hat{k}$. Then the angle between $\vec{a}$ and $\vec{b}=\hat{i}-2\hat{j}+2\hat{k}$ is

(a) $\pi/3$ (b) $\pi/4$ (c) $2\pi/3$ (d) $3\pi/4$

27. If $\vec{a}$ is perpendicular to $\vec{b}$ and p is a non-zero scalar such that $p\vec{r}+(\vec{r}\cdot\vec{b})\vec{a}=\vec{c}$, then

(a) $[\vec{r}\ \vec{a}\ \vec{c}]=0$ (b) $p^2\vec{r}=p\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}$

(c) $p^2\vec{r}=p\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$ (d) $p^2\vec{r}=p\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}$

28. In a four-dimensional space where unit vectors along axes are $\hat{i}, \hat{j}, \hat{k}$ and $\hat{l}$ and $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ are four non-zero vectors such that no vector can be expressed as linear combination of others and $(\lambda-1)(\vec{a}_1-\vec{a}_2)+\mu(\vec{a}_2+\vec{a}_3)+\gamma(\vec{a}_3+\vec{a}_4-2\vec{a}_2)+\vec{a}_3+\delta\vec{a}_4=\vec{0}$ then

(a) $\lambda = 1$ (b) $\mu=-\frac{2}{3}$

(c) $\lambda=\frac{2}{3}$ (d) $\delta=\frac{1}{3}$

29. The equation of a plane is $2x - y - 3z = 5$ and $A(1, 1, 1), B(2, 1, -3), C(1, -2, -2)$ and $D(-3, 1, 2)$ are four points. Which of the following line segments are intersected by the plane?

(a) AD (b) AB (c) AC (d) BC

30. The plane $x - 2y + 7z + 21 = 0$

(a) contains the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$.

(b) contains the point (0, 7, –1).

(c) is perpendicular to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{7}$.

(d) is parallel to the plane $x - 2y + 7z = 0$.

31. If a line makes an angle $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, then

(a) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=1$

(b) $\sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta=\frac{8}{3}$

(c) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$

(d) $\sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta=3$

32. If $OABC$ is a tetrahedron such that $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$, then

(a) $OA \perp BC$ (b) $OB \perp AC$

(c) $OC \perp AB$ (d) $AB \perp AC$

33. If $A(3, 5, -3)$, $B(2, 3, -1)$, $C(1, 2, 3)$ and $D(3, 5, 7)$ are four points, then

(a) AB is parallel to CD.

(b) $AB \perp CD$.

(c) projection of AB on CD is zero.

(d) a, b, c, d are coplanar.

34. If the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ intersects the line

$$3\beta^2 x + 3(1-2\alpha)y + z = 3 = -\frac{1}{2}\{6\alpha^2 x + 3(1-2\beta)y + 2z\},$$

then point $(\alpha, \beta, 1)$ lie on the plane

(a) $2x - y + z = 4$ (b) $x + y - z = 2$

(c) $x - 2y = 0$ (d) $2x - y = 0$

35. Let PM be the perpendicular from the point $P(1, 2, 3)$ to xy plane. If $\overrightarrow{OP}$ makes an angle θ with the positive direction of z-axis and $\overrightarrow{OM}$ makes an angle ϕ with the positive direction of x-axis, where O is the origin and θ and ϕ are acute angles, then

(a) $\tan\theta = \frac{\sqrt{5}}{3}$ (b) $\sin\theta\sin\phi = \frac{2}{\sqrt{14}}$

(c) $\tan\phi = 2$ (d) $\cos\theta\cos\phi = \frac{1}{\sqrt{14}}$

SECTION-III

Comprehension Type

Paragraph for Question No. 36 to 38

Let $\vec{r}$ is a position vector of a variable point in the cartesian plane XOY such that $\vec{r}\cdot(10\hat{j} - 8\hat{i} - \vec{r}) = 40$ and $p_1 = \max\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$, $p_2 = \min\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$. A tangent is drawn to curve $y = \frac{8}{x^2}$ at a point D with abscissa '2' intersect x-axis at E. Then

36. Locus of $\vec{r}$ is

(a) a circle with centre (4, 5)

(b) a circle with ends of diameter (0, 0) and (8, –10)

(c) a circle with radius 1

(d) an ellipse whose major axis is '40'.

37. The value of $p_1 + p_2$ is

(a) $8\sqrt{2}$ (b) 8 (c) $4\sqrt{2}$ (d) 18

38. $\overrightarrow{DE}\cdot\overrightarrow{OE}$ is

(a) 3 (b) $3\sqrt{2}$ (c) $4\sqrt{2}$ (d) 18

Paragraph for Question No. 39 to 41

Let L_1 and L_2 be the lines whose equations are

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \quad \text{and} \quad \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

respectively. A and B are two points on L_1 and L_2 respectively such that AB is perpendicular to both the lines L_1 and L_2.

39. The shortest distance between the lines L_1 and L_2 is

(a) $\sqrt{30}$ (b) $6\sqrt{10}$ (c) $3\sqrt{30}$ (d) $9\sqrt{10}$

40. Coordinates of the point A is

(a) (1, 8, 2) (b) (3, 8, 3)

(c) (–3, 8, 3) (d) none of these

41. Equation of line of shortest distance is

(a) $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$ (b) $\frac{x+3}{2} = \frac{y+7}{5} = \frac{z+6}{1}$

(c) $\frac{x}{2} = \frac{2y-1}{10} = \frac{2z-9}{-1}$ (d) none of these

Paragraph for Question No. 42 to 44

O, A, B are non-collinear points. $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ C is a point such that $\overrightarrow{OC} = 10\vec{a} + 2\vec{b}$. Let Δ_1 be the area of $\square OACB$ and Δ_2 be the area of the parallelogram whose adjacent sides are represented by $\vec{a}$ and $\vec{b}$.

42. An expression for Δ_1 is

(a) $12|\vec{a}\times\vec{b}|$ (b) $3|\vec{a}\times\vec{b}|$

(c) $10|\vec{a}\times\vec{b}|$ (d) $6|\vec{a}\times\vec{b}|$

43. An expression for Δ_2 is

(a) $|\vec{a}\times\vec{b}|$ (b) $\frac{1}{2}|\vec{a}\times\vec{b}|$

(c) $2|\vec{a}\times\vec{b}|$ (d) $4|\vec{a}\times\vec{b}|$

44. $\frac{\Delta_1}{\Delta_2}$ equals

(a) 3 (b) 5 (c) 6 (d) 12

Paragraph for Question No. 45 to 48

Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are forming a right handed system, if $\vec{a}\times\vec{b}=\vec{c}$, $\vec{b}\times\vec{c}=\vec{a}$, $\vec{c}\times\vec{a}=\vec{b}$.

45. If $\vec{x}=\vec{a}+\vec{b}-\vec{c}$, $\vec{y}=-\vec{a}+\vec{b}-2\vec{c}$, $\vec{z}=-\vec{a}+2\vec{b}-\vec{c}$, then a unit vector normal to the vector $\vec{x}+\vec{y}$ and $\vec{y}+\vec{z}$ is

(a) $\vec{a}$ (b) $\vec{b}$

(c) $\vec{c}$ (d) none of these

46. Vectors $2\vec{a}-3\vec{b}+4\vec{c}$, $\vec{a}+2\vec{b}-\vec{c}$ and $x\vec{a}-\vec{b}+2\vec{c}$ are coplanar, then x =

(a) 8/5 (b) 5/8

(c) 0 (d) none of these

47. Let $\vec{x}=\vec{a}+\vec{b}$, $\vec{y}=2\vec{a}-\vec{b}$, then the point of intersection of straight lines $\vec{r}\times\vec{x}=\vec{y}\times\vec{x}$ and $\vec{r}\times\vec{y}=\vec{x}\times\vec{y}$ is

(a) $2\vec{b}$ (b) $3\vec{b}$

(c) $3\vec{a}$ (d) none of these

48. $\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{c}\times\vec{a})+\vec{c}\cdot(\vec{a}\times\vec{b})$ is equal to

(a) 1 (b) 3

(c) 0 (d) none of these

Paragraph for Question No. 49 to 51

Let a plane P_1 passes through the point (1, –2, 3) and is parallel to the plane P_2 given by $2x - 2y + z = 0$.

49. The distance of the point (–1, 2, 0) from the plane P_1 is

(a) 2 units (b) 3 units

(c) 5 units (d) 7 units

50. The coordinate of the foot of perpendicular drawn from point (1, –2, 3) to the plane P_2 is

(a) (0, 0, 0) (b) (–1, 0, 2)

(c) (1, 0, –2) (d) (2, 0, –4)

51. The distance between parallel planes P_1 and P_2 is

(a) 2 units (b) 3 units (c) 5 units (d) 7 units

Paragraph for Question No. 52 and 53

Let $A(2, 3, 5)$, $B(-1, 3, 2)$, $C(\lambda, 5, \mu)$ are the vertices of a triangle and its median through A *i.e,* AD is equally inclined to the coordinate axes.

52. The value of $2\lambda - \mu$ is equal to

(a) 13 (b) 4

(c) 3 (d) none of these

53. Projection of AB on BC is

(a) $\frac{8\sqrt{3}}{11}$ (b) $\frac{-8\sqrt{3}}{\sqrt{11}}$ (c) –48 (d) 48

Paragraph for Question No. 54 to 56

Consider a plane $x + y - z = 1$ and the point $A(1, 2, -3)$. A line L has the equation $x = 1 + 3r$, $y = 2 - r$ and $z = 3 + 4r$.

54. The coordinate of a point B on line L such that AB is parallel to the plane is

(a) (10, –1, 15) (b) (–5, 4, –5)

(c) (4, 1, 7) (d) (–8, 5, –9)

55. The equation of the plane containing the line L and the point A has the equation

(a) $x - 3y + 5 = 0$ (b) $x + 3y - 7 = 0$

(c) $3x - y - 1 = 0$ (d) $3x + y - 5 = 0$

56. The distance between the points on the line which are at a distance of $4/\sqrt{3}$ from the plane is

(a) $4\sqrt{26}$ (b) 20

(c) $10\sqrt{13}$ (d) 30

SECTION-IV

Matrix Match Type

57. Given two vectors $\vec{a}=-\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}+2\hat{k}$.

Column I		Column II	
(A)	A vector coplanar with $\vec{a}$ and $\vec{b}$ is	(p)	$-3\hat{i}+3\hat{j}+4\hat{k}$
(B)	A vector which is perpendicular to both $\vec{a}$ and $\vec{b}$ is	(q)	$2\hat{i}-2\hat{j}+3\hat{k}$
(C)	A vector which is equally inclined to $\vec{a}$ and $\vec{b}$ is	(r)	$\hat{i}+\hat{j}$
(D)	A vector which forms a triangle with $\vec{a}$ and $\vec{b}$ is	(s)	$\hat{i}-\hat{j}+5\hat{k}$

58. Match the following:

	Column I	Column II	
(A)	If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors, then $\lvert\vec{a}-\vec{b}\rvert^2+\lvert\vec{b}-\vec{c}\rvert^2+\lvert\vec{c}-\vec{a}\rvert^2$ does not exceed	(p)	5
(B)	The perpendicular distance from the point (2, 4, –1) to the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$ is	(q)	9
(C)	If $\vec{a}=2\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=x\hat{i}+y\hat{j}+z\hat{k}$, $\vec{a}\cdot\vec{b}=14$ $\vec{a}\times\vec{b}=3\hat{i}+\hat{j}-8\hat{k}$, then $x+y+z=$	(r)	7
		(s)	8

59. Match the following:

	Column I	Column II	
(A)	P is a point on the segment of the line joining (3, 3, 5) and (4, 6, 7) dividing it in the ratio $k:1$ and also projection OP (O, origin) on coordinate axes are $\frac{32}{9},\frac{42}{9},\frac{55}{9}$, then value of k is	(p)	19
(B)	$A(3, 4, 6)$, $B(0, 4, 3)$, $C(a, 6, b)$ are the vertices of a triangle ABC such that the median through A is equally inclined to the coordinate axes, the $a+b$ is	(q)	35
(C)	The line $\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2}$ intersects the curve $yz=k, x=0$, then the value of k is	(r)	15
(D)	Given $\lvert\vec{p}\rvert=2\sqrt{2}, \lvert\vec{q}\rvert=3$. Angle between $\vec{p}$ and $\vec{q}$ is $\pi/4$. If $\vec{a}=\vec{p}-3\vec{q}, \vec{b}=5\vec{p}+2\vec{q}$ form the adjacent sides of a parallelogram then the length of a diagonal is	(s)	12
		(t)	5/4

SECTION-V

Integer Answer Type

60. Let the lines $\vec{\gamma}=(7\hat{i}+6\hat{j}+2\hat{k})+s(-3\hat{i}+2\hat{j}+4\hat{k})$ and $\vec{\gamma}=(5\hat{i}+3\hat{j}+4\hat{k})+t(2\hat{i}+\hat{j}+3\hat{k})$ be intersected by a line parallel to the vector $2\hat{i}-2\hat{j}-\hat{k}$ at P, Q respectively. Then $\lvert\overrightarrow{PQ}\rvert=$

61. If $\vec{a}\times\vec{b}=\vec{c}, \vec{b}\times\vec{c}=\vec{a}, \vec{c}\times\vec{a}=\vec{b}$ and if vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are forming a right handed system, then the volume of tetrahedron formed by vectors $3\vec{a}-2\vec{b}+2\vec{c}$, $-\vec{a}-2\vec{c}$ and $2\vec{a}-3\vec{b}+4\vec{c}$ is

62. The position vectors of the vertices A, B, C of a tetrahedron $ABCD$ are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}$ and $3\hat{i}$ respectively and the altitude from the vertex D to the opposite face ABC meets the face at E. If the length of the edge AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, then the length of DE is

63. If $\vec{a}, \vec{b}$ are vectors perpendicular to each other and $\lvert\vec{a}\rvert=2, \lvert\vec{b}\rvert=3, \vec{c}\times\vec{a}=\vec{b}$, then the least value of $2\lvert\vec{c}-\vec{a}\rvert$ is

64. The square of the distance between the line $\vec{\gamma}=2\hat{i}-2\hat{j}+3\hat{k}+\lambda(\hat{i}+5\hat{j}+\hat{k})$ and the plane $\vec{\gamma}\cdot(\hat{i}-\hat{j}+4\hat{k})=10$, is

65. If $lx + 13y + mz + n = 0$ is the plane through the intersection of the planes $2x + 3y - z + 1 = 0$ and $x + y - 2z + 3 = 0$ and is perpendicular to the plane $3x - y - 2z = 4$ then, $l + m + n =$

SOLUTIONS

1. (d): $\vec{a}\times(\vec{a}\times\vec{b})=\vec{c} \Rightarrow \lvert\vec{a}\rvert\,\lvert\vec{a}\times\vec{b}\rvert=\lvert\vec{c}\rvert$

$\Rightarrow 1(1\times5)\sin\theta=3\Rightarrow\sin\theta=\frac{3}{5}$ gives $\tan\theta=\frac{3}{4}$

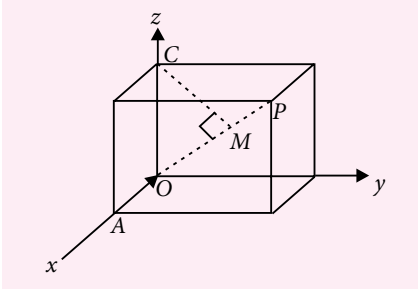
2. (b): From the figure, the vector equation of OP is $\vec{r}=\hat{i}+\hat{j}+\hat{k}$.

$\therefore$ OM = projection of $\overrightarrow{OC}$ on $\overrightarrow{OP}$

$$=\overrightarrow{OC}\cdot\hat{OP}=\hat{k}\cdot\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}$$

Now $CM^2 = OC^2 - OM^2 = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore\ CM = \sqrt{\frac{2}{3}}$



3. (c) : $\vec{p}\times\vec{q}, \vec{p}, \vec{q}$ are non-coplanar vectors.

$\Rightarrow\ b-c=0, c-a=0, a-b=0$

$\Rightarrow\ a=b=c\ \Rightarrow\ \Delta$ is equilateral.

4. (d) : $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

$= -4\vec{c} - 4\vec{d}$

$= -8i \perp$ to both y and z-axes.

5. (d) : Translating the axes through $A(1, 2, 3)$

A changes to $(0, 0, 0)$, B changes to $(8, 6, 2)$

$\therefore$ Coterminous edges are of lengths 8, 6, 2

$\therefore$ Volume of parallelopiped = 8·6·2 = 96 cu.units

6. (d) : The line through $A(1, 2, 3)$, having dr's $(3, 4, 5)$ is $\vec{r} = (1, 2, 3) + t(3, 4, 5)$

Any points on it = $(1 + 3t, 2 + 4t, 3 + 5t)$, t being scalar.

If $(1 + 3t, 2 + 4t, 3 + 5t)$ lies on the plane, then

$1 + 3t + 2(2 + 4t) - 3(3 + 5t) = 5$

$\Rightarrow\ -4t = 9\ \Rightarrow\ t = \frac{-9}{4}$

So the point is $\left(\frac{-23}{4}, \frac{-28}{4}, \frac{-33}{4}\right) = B$

$$\overrightarrow{AB} = \sqrt{\left(1+\frac{23}{4}\right)^2 + \left(2+\frac{28}{4}\right)^2 + \left(3+\frac{33}{4}\right)^2}$$

$$= \frac{1}{4}\sqrt{27^2 + 36^2 + 45^2} = \frac{45\sqrt{2}}{4}$$

7. (d) : Since $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar $\Rightarrow\ [\vec{a}\ \vec{b}\ \vec{c}] \neq 0$

Also $\Rightarrow\ \vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ are non-coplanar. Given, $\vec{d} = \sin x(\vec{a}\times\vec{b}) + \cos y(\vec{b}\times\vec{c}) + 2(\vec{c}\times\vec{a})$. Taking dot product with $\vec{a}+\vec{b}+\vec{c}$, we get

$0 = \sin x\,[\vec{a}\ \vec{b}\ \vec{c}] + \cos y[\vec{a}\ \vec{b}\ \vec{c}] + 2[\vec{a}\ \vec{b}\ \vec{c}]$

$\Rightarrow\ \sin x + \cos y + 2 = 0\ \Rightarrow\ \sin x + \cos y = -2$

$\Rightarrow\ x = (4n-1)\frac{\pi}{2},\ y = (2n-1)\pi,\ n \in Z$

For least value of $x^2 + y^2, x = \frac{-\pi}{2}, y = \pi$ and least value is $\frac{5\pi^2}{4}$.

8. (a) : Cartesian equations of planes are

$\alpha x + \beta y = 0$ ... (1)

$\beta y + \gamma z = 0$ ... (2)

$\gamma z + \alpha x = 0$ ... (3)

$\alpha x + \beta y + \gamma z = p$ ... (4)

(4) cuts (1) at $\left(0, 0, \frac{p}{\gamma}\right)$; (4) cuts (2) at $\left(\frac{p}{\alpha}, 0, 0\right)$

(4) cuts (3) at $\left(0, \frac{p}{\beta}, 0\right)$

(1), (2), (3) are coordinate planes intersecting at (0, 0, 0). So, lengths of the coterminous edges are $\left|\frac{p}{\alpha}\right|, \left|\frac{p}{\beta}\right|, \left|\frac{p}{\gamma}\right|$.

So volume of tetrahedron is $\left|\frac{p^3}{6\alpha\beta\gamma}\right|$ cu.units.

9. (c) : Required plane is passing through (1, 2, 3) and normal to the line joining (1, 2, 3) and (–1, 1, 1)

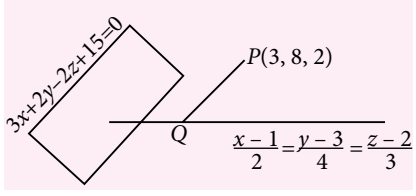
Hence $2(x - 1) + 1(y - 2) + 2(z - 3) = 0$

i.e. $2x + y + 2z = 10$

10. (a) : $Q = (2r + 1, 4r + 3, 3r + 2)$

dr's of $PQ = (2r - 2, 4r - 5, 3r)$

dr's of normal to plane = (3, 2, –2)



Since $PQ \perp$ normal to plane

$\Rightarrow\ 3(2r - 2) + 2(4r - 5) - 2(3r) = 0\ \Rightarrow\ r = 2$

$\therefore\ Q = (5, 11, 8)\ = \therefore$ Required distance $PQ = 7$

11. (d) : Let $Q = (r, r, -r)$ lie on given line such that PQ is perpendicular to $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$

$\therefore\ (\alpha - r)\cdot 1 + (\beta - r)\cdot 1 + (\gamma + r)(-1) = 0$

$\Rightarrow\ r = \dfrac{\alpha + \beta - \gamma}{3}$

$\therefore\ PQ^2 = \dfrac{2}{3}\{\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta + \beta\gamma + \gamma\alpha\}$

but PQ = 2 {$\perp$ distance from $P(\alpha, \beta, \gamma)$ to plane $x + y + z = 0$}

$\Rightarrow\ \dfrac{2}{3}(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta + \beta\gamma + \gamma\alpha) = 4\left\{\dfrac{\alpha + \beta + \gamma}{\sqrt{3}}\right\}^2$

$\therefore\ \alpha^2 + \beta^2 + \gamma^2 + 5\alpha\beta + 3\beta\gamma + 3\gamma\alpha = 0$

12. (b) : Equation of given plane is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

Given, $a^2 + b^2 + c^2 = 9$

dr's of normal to the plane is $\left(\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}\right)$

Let P be the foot of the perpendicular then
dr's of $OP = (x_1, y_1, z_1)$
OP is parallel to normal

$\Rightarrow\ \dfrac{x_1}{1/a} = \dfrac{y_1}{1/b} = \dfrac{z_1}{1/c} = k \Rightarrow\ a = \dfrac{k}{x_1}, b = \dfrac{k}{y_1}, c = \dfrac{k}{z_1}$

Since $a^2 + b^2 + c^2 = 9 \Rightarrow k^2\left(\dfrac{1}{x_1^2} + \dfrac{1}{y_1^2} + \dfrac{1}{z_1^2}\right) = 9$ ... (i)

By substituting $a = \dfrac{k}{x_1}, b = \dfrac{k}{y_1}, c = \dfrac{k}{z_1}$ in

$\dfrac{x_1}{a} + \dfrac{y_1}{b} + \dfrac{z_1}{c} = 1$, we get $x_1^2 + y_1^2 + z_1^2 = k$

From (i), $(x_1^2 + y_1^2 + z_1^2)^2(x_1^{-2} + y_1^{-2} + z_1^{-2}) = 9$

13. (d) : Suppose M divides AC in the ratio $\lambda : 1$. Then

$$M = \left(\frac{-4\lambda + 1}{\lambda + 1}, \frac{\lambda - 2}{\lambda + 1}, \frac{\lambda + 2}{\lambda + 1}\right)$$

Hence $\overrightarrow{BM} = \left(\dfrac{-5\lambda}{\lambda + 1}, \dfrac{-3\lambda - 6}{\lambda + 1}, \dfrac{\lambda + 2}{\lambda + 1}\right)$

and $\overrightarrow{AC} = (-5, 3, -1)$. Now $\overrightarrow{BM} \perp \overrightarrow{AC}$

$\Rightarrow\ 25\lambda - 9\lambda - 18 - \lambda - 2 = 0$

$\Rightarrow\ 15\lambda = 20 \Rightarrow \lambda = \dfrac{4}{3}$

$\therefore\ \overrightarrow{BM} = \left(\dfrac{-20}{7}, \dfrac{-30}{7}, \dfrac{10}{7}\right)$

14. (b) : Direction ratios of line I is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \hat{i}(-5) - \hat{j}(-7) + \hat{k}(1) = (-5, 7, 1)$$

Direction ratios of line II is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 4 \\ 1 & 1 & -4 \end{vmatrix} = \hat{i}(0) - \hat{j}(-8) + \hat{k}(2) = (0, 8, 2)$$

Let θ be angle between the lines, then

$$\cos\theta = \frac{(-5\times 0) + (7\times 8) + (1\times 2)}{\sqrt{25 + 49 + 1}\sqrt{0 + 64 + 4}}$$

$$\Rightarrow\ \theta = \cos^{-1}\left\{\frac{58}{(5\sqrt{3})(2\sqrt{17})}\right\} = \cos^{-1}\left\{\frac{29}{5\sqrt{51}}\right\}$$

15. (d) : If P is a variable point on the required plane, then the equation of the plane is $[\overrightarrow{AP}\ \overrightarrow{AB}\ \overrightarrow{AB} \times \overrightarrow{AC}] = 0$

16. (c) : Here P and Q lie on the same side of XY plane.
Image of $P(1, 2, 3)$ on the XY plane is $P'(1, 2, -3)$

Reflected ray $P'Q$ is $\dfrac{x - 3}{2} = \dfrac{y - 2}{0} = \dfrac{z - 5}{8}$

$$\Rightarrow\ \frac{x - 3}{1} = \frac{y - 2}{0} = \frac{z - 5}{4}$$

17. (c) : Observe that the distance from the origin to the plane is invariant, since the origin and the plane are fixed.

18. (b) : Required distance

$$= \left|\frac{ax_1 + by_1 + cz_1 + d}{a\cos\alpha + b\cos\beta + c\cos\gamma}\right|$$

$$= \left|\frac{3 + 4 + 5 - 2}{\frac{1}{3} + \frac{2}{3} + \frac{2}{3}}\right| = 6$$

Using $(\cos\alpha, \cos\beta, \cos\gamma) = \left(\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right)$

19. (a) : Point $(1, -1, 3)$ lies on plane, $x - 2y - z = 0$. So l lines in P.

20. (a) : $P(a, b, -c)$; $Q(-a, b, c)$; $R(a, -b, c)$

$$\Rightarrow\ G = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

$\therefore$ OAG are collinear.

21. (a) : Equation of plane through (1, 0, –1) will be $a(x-1)+b(y-0)+c(z+1)$ passing through (3, 2, 2) gives $2a+2b+3c=0$ and parallel to given line gives $a-2b+3c=0 \Rightarrow \frac{a}{4}=\frac{b}{-1}=\frac{c}{-2}$

$\therefore$ Equation of plane will be $4x-y-2z-6=0$

22. (a, b) : We have,

$$\alpha(\vec{a}\times\vec{b})+\beta(\vec{b}\times\vec{c})+\gamma(\vec{c}\times\vec{a})=0$$

Taking dot product with $\vec{c}$, we have

$$\alpha[\vec{a}\ \vec{b}\ \vec{c}]+\beta[\vec{b}\ \vec{c}\ \vec{c}]+\gamma[\vec{c}\ \vec{a}\ \vec{c}]=0$$

i.e. $\alpha[\vec{a}\ \vec{b}\ \vec{c}]+0+0=0 \Rightarrow \alpha[\vec{a}\ \vec{b}\ \vec{c}]=0$

Similarly, taking dot product with $\vec{b}$ and $\vec{a}$, we have

$$\gamma[\vec{a}\ \vec{b}\ \vec{c}]=0, \beta[\vec{a}\ \vec{b}\ \vec{c}]=0$$

Now, even if one of $\alpha, \beta, \gamma \neq 0$, then we have $[\vec{a}\ \vec{b}\ \vec{c}]=0$

$\Rightarrow$ $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

23. (b, c, d) :

(a) $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$

$\Rightarrow$ vectors are coplanar, so do not form right handed system.

(b) $(\vec{a}\times\vec{b})\times\vec{c}$, $\vec{a}\times\vec{b}$, $\vec{c}$ in that order form right handed system.

$\Rightarrow$ $\vec{c}$, $(\vec{a}\times\vec{b})\times\vec{c}, \vec{a}\times\vec{b}$ also form right handed system as they are in same cyclic order.

(c) $\vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow (\vec{a}+\vec{b}+\vec{c})^2=0$

$\Rightarrow$ $\vec{a}^2+\vec{b}^2+\vec{c}^2=-2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

Hence $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}<0$

(d) $\vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow \vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

Using this we get result.

24. (a,b,c,d) : Since $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$ are non - coplanar, $\vec{r}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})$

$\vec{r}\times\vec{b}=\vec{a} \Rightarrow x(\vec{a}\times\vec{b})+z(\vec{a}\times\vec{b})\times\vec{b}=\vec{a}$

$\Rightarrow$ $x(\vec{a}\times\vec{b})+z\{(\vec{b}\cdot\vec{a})\vec{b}-(\vec{b}\cdot\vec{b})\vec{a}\}=\vec{a}$

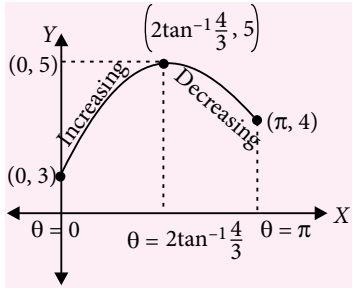
$\Rightarrow$ $x(\vec{a}\times\vec{b})-\vec{a}(1+|\vec{b}|^2 z)=0$

$\Rightarrow$ $x=0, z=-\frac{1}{|\vec{b}|^2}$

$\therefore$ $\vec{r}=y\vec{b}-\frac{1}{|\vec{b}|^2}(\vec{a}\times\vec{b})$, where y is any scalar.

25. (b, c, d) : Let angle between $\vec{a}$ and $\vec{b}$ be θ.

We have, $|\vec{a}|=|\vec{b}|=1$



Now, $|\vec{a}+\vec{b}|=2\cos\frac{\theta}{2}$ and $|\vec{a}-\vec{b}|=2\sin\frac{\theta}{2}$

Consider $F(\theta)=\frac{3}{2}\left(2\cos\frac{\theta}{2}\right)+2\left(2\sin\frac{\theta}{2}\right)$

$\therefore$ $F(\theta)=3\cos\frac{\theta}{2}+4\sin\frac{\theta}{2}, \theta\in[0,\pi]$

$\therefore$ $F'(\theta)=\frac{-3}{2}\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}$

Now, $F'(\theta)=0 \Rightarrow \tan\frac{\theta}{2}=\frac{4}{3}$

Clearly, $F(0)=3$

$$F\left(\theta=2\tan^{-1}\frac{4}{3}\right)=3\left(\frac{3}{5}\right)+4\left(\frac{4}{5}\right)$$

$$=\frac{9}{5}+\frac{16}{5}=\frac{25}{5}=5$$

$F(\theta=\pi)=4$. $\therefore$ Range = [3, 5]

Hence possible integer(s) in the range of $F(\theta)$ in $[0, \pi]$ are 3. *viz.* 3, 4 and 5.

26. (b, d) : Let $\vec{r}=(x, y, z)$ be a point in the plane containing $\hat{i}$ and $\hat{i}+\hat{j}$ then $\vec{r}-\hat{i}, \hat{i}, \hat{i}+\hat{j}$ will be coplanar *i.e.* $z=0$

The equation of plane containing $\hat{i}-\hat{j}$ and $\hat{i}+\hat{k}$ is

$$x+y-z=0$$

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$a_3=0, a_1+a_2=0$

$\therefore$ $\vec{a}=a_1(\hat{i}-\hat{j}) \Rightarrow \cos\theta=\pm\frac{1}{\sqrt{2}}$

27. (a, d) : Let $\vec{r}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})$.

Taking dot product with $\vec{b}$

$\Rightarrow$ $\vec{r}\cdot\vec{b}=0+y|\vec{b}|^2$; $\vec{a}(\vec{r}\cdot\vec{b})=y(\vec{b})^2\vec{a}$

$\Rightarrow \vec{c} - p\vec{r} = y|\vec{b}|^2\vec{a} \Rightarrow \vec{r} = \frac{1}{p}\vec{c} - \frac{y|\vec{b}|^2}{p}\vec{a}$

$\therefore [\vec{r}\ \vec{a}\ \vec{c}] = 0$. Now $\vec{r}\cdot\vec{b} = \frac{1}{p}\vec{c}\cdot\vec{b}$

$\therefore y|\vec{b}|^2 = \frac{\vec{b}\cdot\vec{c}}{p} \quad \therefore \vec{r} = \frac{1}{p}\vec{c} - \frac{1}{p^2}(\vec{b}\cdot\vec{c})\vec{a}$

28. (a, b, d) : $(\lambda-1)(\vec{a}_1-\vec{a}_2)+\mu(\vec{a}_2+\vec{a}_3)$
$+\gamma(\vec{a}_3+\vec{a}_4-2\vec{a}_2)+\vec{a}_3+\delta\vec{a}_4=\vec{0}$

i.e. $(\lambda-1)\vec{a}_1+(1-\lambda+\mu-2\gamma)\vec{a}_2+(\mu+\gamma+1)\vec{a}_3$
$+(\gamma+\delta)\vec{a}_4=\vec{0}$

Since $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ are linearly independent.

$\therefore$ $\lambda - 1 = 0, 1 - \lambda + \mu - 2\gamma = 0, \mu + \gamma + 1 = 0,$
$\gamma + \delta = 0$

i.e. $\lambda = 1, \mu + \gamma + 1 = 0, \gamma + \delta = 0$

i.e. $\lambda = 1, \mu = -\frac{2}{3}, \gamma = -\frac{1}{3}, \delta = \frac{1}{3}$

29. (b, c) : For $A(1, 1, 1)$,
$2x - y - 3z - 5 = 2 - 1 - 3 - 5 < 0$

For $B(2, 1, -3)$, $2x - y - 3z - 5 = 4 - 1 + 9 - 5 > 0$

For $C(1, -2, -2)$, $2x - y - 3z - 5 = 2 + 2 + 6 - 5 > 0$

For $D(-3, 1, 2)$, $2x - y - 3z - 5 = -6 - 1 - 6 - 5 < 0$

$\therefore$ A, D are on one side of the plane and B, C are on the other side.

$\therefore$ The line segments AB, AC, BD, CD intersect the plane.

30. (a, b, c, d) : Conditions for the plane $ax + by + cz + d = 0$ to contain the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ are $a\alpha + b\beta + c\gamma + d = 0$ and $al + bm + cn = 0$.

Here the condition is satisfied for option (a). For option (c), normal to plane and given line have same d.r's.

$\therefore$ The line is $\perp$ to the given plane.

For option (b), clearly the point satisfies the given plane.

The d.r's of normal to the plane in option (d) are $(1, -2, 7)$ which are same as the d.r's of normal to the given plane.

$\therefore$ The plane given in option (d) is parallel to given plane.

31. (b, c) : d.c's of diagonals are
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),$
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

Let $<l, m, n>$ be the d.c's of line which makes an angle $\alpha, \beta, \gamma, \delta$ with the diagonals, then

$\cos\alpha = \frac{l+m+n}{\sqrt{3}}, \cos\beta = \frac{-l+m+n}{\sqrt{3}},$

$\cos\gamma = \frac{l-m+n}{\sqrt{3}}, \cos\delta = \frac{l+m-n}{\sqrt{3}}$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = \frac{8}{3}$

32. (a, b, c) : Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$, then
$\vec{a}\cdot\vec{a} + (\vec{b}-\vec{c})\cdot(\vec{b}-\vec{c}) = \vec{b}\cdot\vec{b} + (\vec{c}-\vec{a})\cdot(\vec{c}-\vec{a})$

$\Rightarrow -2\vec{b}\cdot\vec{c} = -2\vec{c}\cdot\vec{a} \Rightarrow (\vec{a}-\vec{b})\cdot\vec{c} = 0$

or $\overrightarrow{BA}\cdot\overrightarrow{OC} = 0$. Hence $\overrightarrow{AB} \perp \overrightarrow{OC}$

Similarly $\overrightarrow{BC} \perp \overrightarrow{OA}$ and $\overrightarrow{CA} \perp \overrightarrow{OB}$

33. (b, c) : d.r's of $AB = (-1, -2, 2)$
d.r's of $CD = (2, 3, 4)$. $\therefore$ $AB \perp CD$

d.c's of $CD = \left(\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)$

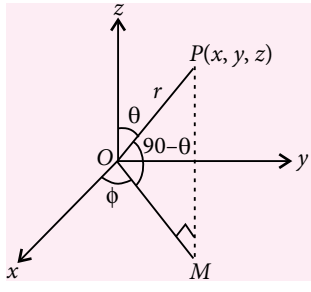
$\therefore$ Projection of AB on CD is zero.

34. (a, b, c) : Intersection of line with both the planes are same

$\Rightarrow \frac{3}{3\beta^2 + 6(1-2\alpha) + 3} = \frac{-6}{6\alpha^2 + 6(1-2\beta) + 6}$

$\Rightarrow 3(\beta-1)^2 + 3(\alpha-2)^2 = 0 \Rightarrow \alpha = 2, \beta = 1.$

35. (a, b, c) :



If P be (x, y, z) then from the figure,

$OM = r\cos(90 - \theta) = r\sin\theta$

$\Rightarrow$ $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$\Rightarrow$ $1 = r\sin\theta\cos\phi$, $2 = r\sin\theta\sin\phi$, $3 = r\cos\theta$

$\Rightarrow$ $1^2 + 2^2 + 3^2 = r^2 \Rightarrow r = \pm\sqrt{14}$

$\therefore$ $\sin\theta\cos\phi = \frac{1}{\sqrt{14}}$, $\sin\theta\sin\phi = \frac{2}{\sqrt{14}}$, $\cos\theta = \frac{3}{\sqrt{14}}$

(Neglecting negative sign as θ and ϕ are acute).

$\therefore$ $\frac{\sin\theta\sin\phi}{\sin\theta\cos\phi} = \frac{2}{1} \Rightarrow \tan\phi = 2$.

Also, $\tan\theta = \frac{\sqrt{5}}{3}$.

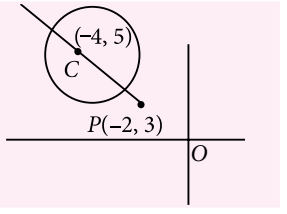
36. (c) : $\vec{r} = x\hat{i} + y\hat{j}$ (XOY plane)

We have, $\vec{r}.(10\hat{j} - 8\hat{i} - \vec{r}) = 40$

$\Rightarrow$ $x^2 + y^2 + 8x - 10y + 40 = 0$

Centre $(-4, 5)$, radius $= \sqrt{16 + 25 - 40} = 1$

37. (d) : $p_1 = (CP + r)^2$, $p_2 = (CP - r)^2$



$p_1 = (2\sqrt{2} + 1)^2$, $p_2 = (2\sqrt{2} - 1)^2$ $\Rightarrow p_1 + p_2 = 18$

38. (a) : Equation of tangent at $D(2, 2)$ to $y = 8/x^2$ is $2x + y = 6$

$\therefore$ $E = (3, 0) \Rightarrow \overrightarrow{OE} = 3\hat{i}$ ($\because$ point E lies on X-axis)

$\overrightarrow{OD} = 2\hat{i} + 2\hat{j}$; $\overrightarrow{DE} = \hat{i} - 2\hat{j}$; $\overrightarrow{DE}\cdot\overrightarrow{OE} = 3$.

39. (c) : Let $\vec{a} = (3, 8, 3)$, $\vec{c} = (-3, -7, 6)$, $\vec{b} = 3\hat{i} - \hat{j} + \hat{k}$, $\vec{d} = -3\hat{i} + 2\hat{j} + 4\hat{k}$

Shortest distance between L_1, L_2 is

$$\left|\frac{(\vec{c} - \vec{a})\cdot(\vec{b}\times\vec{d})}{|\vec{b}\times\vec{d}|}\right| = \frac{270}{\left|-6\hat{i} - 15\hat{j} + 3\hat{k}\right|} = \frac{270}{\sqrt{270}} = \sqrt{270}$$

40. (b) : Notice that line joining the points (3, 8, 3) and (–3, –7, 6) is line with d.r's (6, 15, –3) & is perpendicular to both L_1 & L_2.

So take $A = (3, 8, 3)$, $B = (-3, -7, 6)$

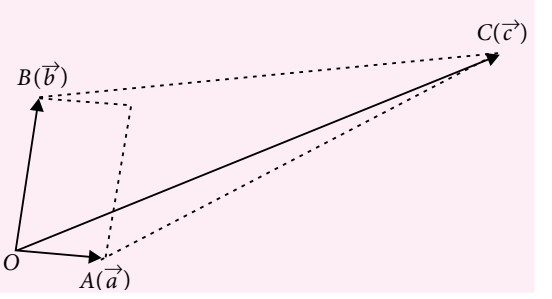
41. (a) : Line of shortest distance is line joining AB only in this case (given in passage)

As d.r's of AB are (6, 15, –3) or (2, 5, –1)

$\therefore$ Equation of line AB passing through (3, 8, 3) is

$$\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

42. (d) :



$$\Delta_1 = \frac{1}{2}\,|\overrightarrow{BA}\times\overrightarrow{OC}|$$

$$= \frac{1}{2}\left|(\vec{a} - \vec{b})\times(10\vec{a} + 2\vec{b})\right| = 6\,|\vec{a}\times\vec{b}|$$

43. (a) : $\Delta_2 = |\vec{a}\times\vec{b}|$

44. (c) : $\frac{\Delta_1}{\Delta_2} = \frac{6|\vec{a}\times\vec{b}|}{|\vec{a}\times\vec{b}|} = 6$

45. (d) : $\vec{x} + \vec{y} = 2\vec{b} - 3\vec{c}$ and $\vec{y} + \vec{z} = -2\vec{a} + 3\vec{b} - 3\vec{c}$

$$\therefore\ (\vec{x} + \vec{y})\times(\vec{y} + \vec{z}) = \begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ 0 & 2 & -3 \\ -2 & 3 & -3 \end{vmatrix} = 3\vec{a} + 6\vec{b} + 4\vec{c}$$

$\therefore$ Required unit vector $= \frac{3\vec{a} + 6\vec{b} + 4\vec{c}}{\sqrt{61}}$

46. (a) : $\begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ x & -1 & 2 \end{vmatrix} = 0$

$\Rightarrow$ $2(4 - 1) + 3(2 + x) + 4(-1 - 2x) = 0 \Rightarrow x = \frac{8}{5}$

47. (c) :

$\vec{r}\times\vec{x} = \vec{y}\times\vec{x} \Rightarrow (\vec{r} - \vec{y})\times\vec{x} = 0 \Rightarrow \vec{r} = \vec{y} + \lambda\vec{x}$

$\vec{r}\times\vec{y} = \vec{x}\times\vec{y} \Rightarrow (\vec{r} - \vec{x})\times\vec{y} = 0 \Rightarrow \vec{r} = \vec{x} + \mu\vec{y}$

$\Rightarrow$ $\vec{y} + \lambda\vec{x} = \vec{x} + \mu\vec{y}$

$\Rightarrow$ $(2\vec{a} - \vec{b}) + \lambda(\vec{a} + \vec{b}) = (\vec{a} + \vec{b}) + \mu(2\vec{a} - \vec{b})$

$\Rightarrow$ $2 + \lambda = 1 + 2\mu$, $-1 + \lambda = 1 - \mu \Rightarrow \mu = 1, \lambda = 1$

$\therefore$ The point of intersection is $3\vec{a}$.

48. (b) : $\vec{a}\times\vec{b} = \vec{c} \Rightarrow \vec{c}\cdot(\vec{a}\times\vec{b}) = \vec{c}\cdot\vec{c} = 1 \Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 1$

$\therefore$ $\vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{c}\times\vec{a}) + \vec{c}\cdot(\vec{a}\times\vec{b}) = 3$

49. (c) 50. (b) 51. (b)

The equation of the plane P_1 is $2x - 2y + z = \lambda$

Since, it passes through $(1, -2, 3)$ $\therefore$ $\lambda = 9$

So, P_1 is $2x - 2y + z = 9$

Its distance from point $(-1, 2, 0)$ is

$$\frac{|2\times(-1) - 2\times(2) + 0 - 9|}{3} = 5 \text{ units}$$

Now, the line perpendicular to plane P_2 and passing through $(1, -2, 3)$ is given by

$$\frac{x-1}{2} = \frac{y+2}{-2} = \frac{z-3}{1}$$

Any point on this line can be taken as

$(2t + 1, -2t - 2, t + 3)$

If it lies on plane P_2 then, we have

$2(2t + 1) - 2(-2t - 2) + t + 3 = 0$

$\Rightarrow$ $9t + 9 = 0$ $\Rightarrow$ $t = -1$

So, the coordinate of the foot of perpendicular drawn from point $(1, -2, 3)$ to the plane P_2 is $(-1, 0, 2)$.

Again, the distance between the parallel planes $2x - 2y + z - 9 = 0$ and $2x - 2y + z = 0$ is given by

$$\frac{|9-0|}{\sqrt{2^2 + (-2)^2 + 1}} = \frac{9}{3} = 3 \text{ units}$$

52. (b) : Mid point of BC, $D = \left(\frac{\lambda-1}{2}, 4, \frac{\mu+2}{2}\right)$

d.r's of $AD = \left(\frac{\lambda-5}{2}, 1, \frac{\mu-8}{2}\right)$

Since AD is equally inclined to axes

$\Rightarrow$ $\lambda = 7, \mu = 10 \Rightarrow 2\lambda - \mu = 4$

53. (b) : $A(2, 3, 5)$, $B(-1, 3, 2)$ $C(7, 5, 10)$

$\therefore$ $\overrightarrow{AB} = -3\hat{i} - 3\hat{k}$, $\overrightarrow{BC} = 8\hat{i} + 2\hat{j} + 8\hat{k}$

Projection of $\overrightarrow{AB}$ on $\overrightarrow{BC} = \frac{\overrightarrow{AB}\cdot\overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{-8\sqrt{3}}{\sqrt{11}}$

54. (d) : The given line is $\frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{4} = r$

Any point say $B \equiv (3r + 1, 2 - r, 3 + 4r)$ (on the line L)

$\overrightarrow{AB} = (3r, -r, 4r + 6)$

Since $\overrightarrow{AB}$ is parallel to $x + y - z = 1$.

$\Rightarrow$ $3r - r - 4r - 6 = 0 \Rightarrow r = -3$

$\therefore$ $B = (-8, 5, -9)$

55. (b) : The equation of plane containing the line L is $a(x - 1) + b(y - 2) + c(z - 3) = 0$....(i), where $3a - b + 4c = 0$

Since (i), also contains the point $A(1, 2, -3)$.

Hence $c = 0$ and $3a = b$

$\therefore$ The equation of the plane is $x - 1 + 3(y - 2) = 0$, or $x + 3y - 7 = 0$

56. (a) : The distance of point $(1 + 3r, 2 - r, 3 + 4r)$ from the plane is

$$\frac{|1+3r+2-r-3-4r|}{\sqrt{1+1+1}} = \frac{|2r|}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

$\Rightarrow$ $r = \pm 2$

Hence, the points are $A(7, 0, 11)$ and $B(-5, 4, -5)$

$\Rightarrow$ $AB = \sqrt{416} = 4\sqrt{26}$

57. (A → r; B → q; C → s; D → p)

(A) Vector $\hat{i} + \hat{j}$ is coplanar with $\vec{a}$ and $\vec{b}$.

(B) $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ -2 & 1 & 2 \end{vmatrix} = 2\hat{i} - 2\hat{j} + 3\hat{k}$

(C) If $\vec{c}$ is equally inclined to $\vec{a}$ and $\vec{b}$, then we must have $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$, which is true for $\vec{c} = \hat{i} - \hat{j} + 5\hat{k}$

(D) If vector $\vec{c}$ is forming a triangle with $\vec{a}$ and $\vec{b}$. Then

$\vec{c} = \vec{a} + \vec{b} = -3\hat{i} + 3\hat{j} + 4\hat{k}$.

58. (A → q; B → r; C → s)

(A) $(\vec{a} + \vec{b} + \vec{c})^2 \geq 0$ $\Rightarrow$ $-2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) \leq 3$

$\Rightarrow$ $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 \leq 2(1 + 1 + 1) + 3 = 9$

(B) $P = (-5, -3, 6)$, $Q = (2, 4, -1)$, $PQ = \sqrt{147}$ and Projection of $PQ = \sqrt{98}$ $\therefore$ $QM = \sqrt{147 - 98} = 7$

(C) $\vec{a}\times(\vec{a}\times\vec{b}) = -17\hat{i} + 19\hat{j} - 4\hat{k}$

$\Rightarrow$ $(\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{a})\vec{b} = -17\hat{i} + 19\hat{j} - 4\hat{k}$

$\Rightarrow$ $14\vec{a} - 9\vec{b} = -17\hat{i} + 19\hat{j} - 4\hat{k}$

$\Rightarrow$ $\vec{b} = 5\hat{i} + \hat{j} + 2\hat{k} \Rightarrow x + y + z = 8$

59. (A → t; B → p; C → q; D → r)

(A) $\overrightarrow{OP} = \frac{32}{9}\hat{i} + \frac{42}{9}\hat{j} + \frac{55}{9}\hat{k}$

$\therefore$ $\frac{4k+3}{k+1} = \frac{32}{9}, \frac{6k+3}{k+1} = \frac{42}{9}, \frac{7k+5}{k+1} = \frac{55}{9}$

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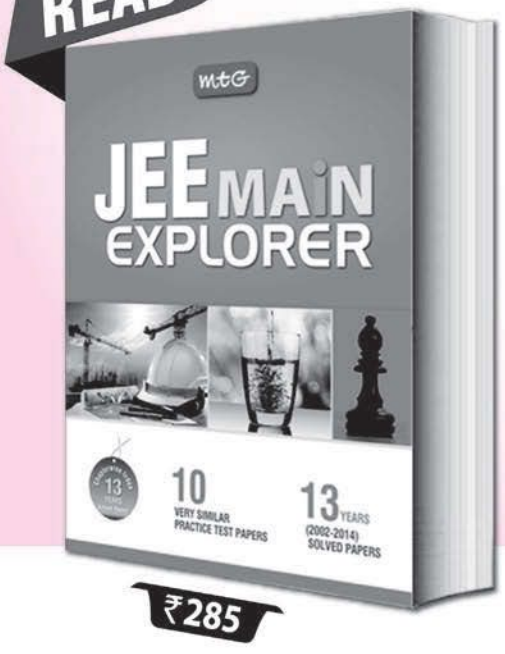

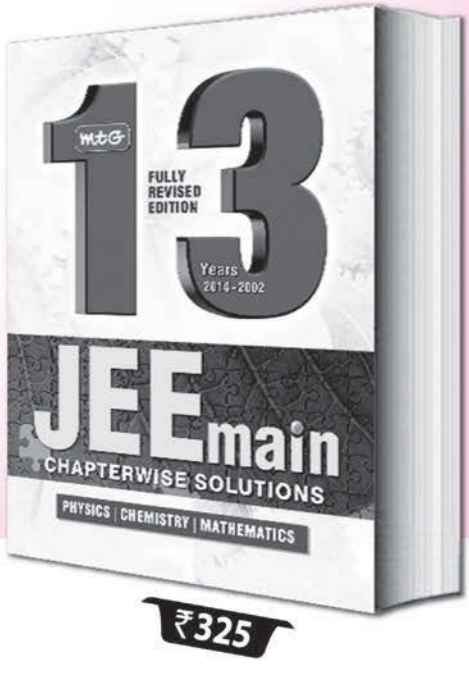

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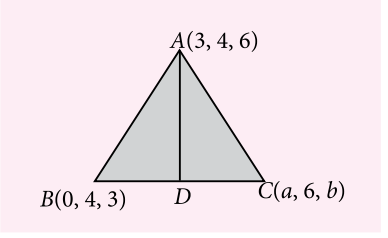
All these gives $k=\frac{5}{4}$

(B) Mid point D of $BC=\left(\frac{a}{2},5,\frac{b+3}{2}\right)$

Direction ratios of median AD are

$$\left(\left(3-\frac{a}{2}\right),-1,\ 6-\left(\frac{b+3}{2}\right)\right)$$

This is equally inclined to the axes.



$$\therefore\quad 3-\frac{a}{2}=-1,\quad 6-\left(\frac{b+3}{2}\right)=-1$$

$$\Rightarrow\ a=8,\ b=11\ \Rightarrow\ a+b=19$$

(C) Solve with $x=0$

$$\frac{y-12}{5}=-1\Rightarrow y=7,\ \frac{z-7}{2}=-1\ \Rightarrow\ z=5$$

$$yz=k\ \Rightarrow\ k=35$$

(D) One diagonal is given by $\vec{a}+\vec{b}=6\vec{p}-\vec{q}$

$$(\text{Length})^2=36\,|\vec{p}|^2+|\vec{q}|^2-12.|\vec{p}||\vec{q}|\cos\frac{\pi}{4}=225$$

$$\text{Length}=\sqrt{225}=15$$

$$\therefore\quad \text{Other diagonal}=4\vec{p}+5\vec{q}$$

$$(\text{Length})^2=(16\times 8)+(25\times 9)+40\cdot 2\sqrt{2}\cdot 3\cdot\frac{1}{\sqrt{2}}=593$$

$$\text{Length}=\sqrt{593}$$

60. (9) : Let $P=(7-3s,\ 6+2s,\ 2+4s)$

$Q=(5+2t,\ 3+t,\ 4+3t)$

PQ is parallel to $\vec{r}=(2\hat{i}-2\hat{j}-\hat{k})$

i.e. $\overrightarrow{PQ}=\lambda(2\hat{i}-2\hat{j}-\hat{k})$

$$\overrightarrow{PQ}=OQ-OP=(-2+2t+3s,\ -3+t-2s,\ 2+3t-4s)$$

$\therefore\quad -2+2t+3s=2\lambda;\ -3+t-2s=-2\lambda;$

$2+3t-4s=-\lambda$

Solving, we get $\lambda=3,\quad\therefore\quad \left|\overrightarrow{PQ}\right|=3\left|2\hat{i}-2\hat{j}-\hat{k}\right|=9$

61. (2) : $|\vec{a}|=|\vec{b}|=|\vec{c}|=1;\ [\vec{a}\,\vec{b}\,\vec{c}]=1$

Volume of the tetrahedron $=\frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$

$$=\frac{1}{6}\begin{vmatrix}3&-2&2\\-1&0&-2\\2&-3&4\end{vmatrix}=2$$

62. (2) : $\overrightarrow{BC}$ is the x-axis and $\overrightarrow{AB}\perp\overrightarrow{BC}$

Area of $\Delta ABC=\frac{1}{2}\times\sqrt{2}\times 2=\sqrt{2}$

$$\text{Volume}=\frac{1}{3}\times\sqrt{2}\times DE=\frac{2\sqrt{2}}{3}\Rightarrow DE=2$$

63. (3) : $\vec{c}\times\vec{a}=\vec{b}\Rightarrow|\vec{c}\times\vec{a}|=|\vec{b}|\Rightarrow|\vec{c}||\vec{a}|\sin\theta=3,$

$$\Rightarrow\quad |\vec{c}|=\frac{3}{2\sin\theta}\Rightarrow\ |\vec{c}-\vec{a}|^2=|\vec{c}|^2+|\vec{a}|^2-2\vec{c}\cdot\vec{a}$$

$$=|\vec{c}|^2+4-2|\vec{c}|\cdot|\vec{a}|\cos\theta$$

$$=\frac{9}{4\sin^2\theta}+4-2\cdot\frac{3}{2\sin\theta}\cdot 2\cdot\cos\theta$$

$$=4+\frac{9}{4}\text{cosec}^2\theta-6\cot\theta=\frac{9}{4}+\left(\frac{3}{2}\cot\theta-2\right)^2$$

$$\Rightarrow\quad |\vec{c}-\vec{a}|^2\ge\frac{9}{4}\Rightarrow|\vec{c}-\vec{a}|\ge\frac{3}{2}$$

$$\Rightarrow\quad 2|\vec{c}-\vec{a}|\ge 3\ \therefore\ \text{Min. of } 2|\vec{c}-\vec{a}|=3$$

64. (2) : See that line is $\perp^r$ to plane as

$$(\hat{i}+5\hat{j}+\hat{k})\cdot\left(\hat{i}-\hat{j}+4\hat{k}\right)=1-5+4=0$$

So, distance between line & plane = distance from $(2, -2, 3)$ to the plane $x - y + 4z = 10$

$$=\frac{|2+2+12-10|}{\sqrt{1+1+16}}=\sqrt{2}$$

$\therefore$ Square of the distance = 2

65 (2) : The given plane will be of the form $2x + 3y - z + 1 + \lambda(x + y - 2z + 3) = 0$. If this is perpendicular to $3x - y - 2z - 4 = 0$, then

$3(\lambda + 2) + (-1)(3 + \lambda) + (-2)(-1 - 2\lambda) = 0$

$\Rightarrow\ 6\lambda + 5 = 0\ \Rightarrow \lambda = -5/6$

$\therefore$ The plane is $7x + 13y + 4z - 9 = 0$

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Series-8 **Linear Programming & Probability**

LINEAR PROGRAMMING

OPTIMISATION PROBLEM

A problem in which a linear function is to be optimised (maximise or minimise) satisfying certain linear inequalities is called optimisation problem.

LINEAR PROGRAMMING PROBLEM (L.P.P.)

It is a method of determining a particular program or plane of action to optimise a linear function of two variables (say x and y) known as objective function subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints).

FORMATION OF LINEAR PROGRAMMING PROBLEM (L.P.P.)

Let $x_1, x_2, x_3, ..., x_n$ be n variables. Let the objective function to be optimised (*i.e.*, minimised or maximised) is given by Z.

(i) $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$, where $c_i's (i = 1, 2, ..., n)$ are constants.

(ii) Let there be mn constants and let b_i be a set of constants such that

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n$ $(\leq, =$ or $\geq) b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n$ $(\leq, =$ or $\geq) b_2$

....

....

$a_{ml}x_1 + a_{m2}x_2 + ... + a_{mn}x_n$ $(\leq, =$ or $\geq) b_m$

(iii) Finally, let $x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$, are called non-negative constraints.

The problem of determining the values of $x_1, x_2, ..., x_n$ which makes Z, a minimum or maximum and which satisfies (ii) and (iii) is called the general linear programming problem.

TERMS USED IN L.P.P.

- **Objective function :** The linear function which is to be optimised (maximised or minimised) is called linear objective function.

  For example, linear function $Z = ax + by$, where a and b are constants is a linear objective function.
- **Decision variables :** The variables that enter into the problem are called decision variables. In the objective function $Z = ax + by$, x and y are called decision variables.
- **Linear constraints :** The linear inequations (inequalities) or restrictions on the decision variables of a linear programming problem is called linear constraints. The conditions $x \geq 0$, $y \geq 0$ are called non-negative constraints.
- **Convex region :** A region or a set of points is said to be convex if the line joining any two of its points lies completely in the region.
- **Feasible region :** The common region determined by all the constraints including non-negative constraints of a L.P.P. is called the feasible region or solution region. Feasible region is always a convex set. Feasible region may be bounded or unbounded.
- **Bounded and unbounded feasible regions :** A feasible region of a system of linear inequations is said to be bounded if it can be enclosed within a circle. But if the feasible region extends indefinitely in any direction, then the feasible region is called unbounded.

- **Corner points of a feasible region :** A point in the region which is the intersection of two boundary lines are called corner points.
- **Feasible solutions :** Points within and on the boundary of the feasible region of a L.P.P. represent feasible solutions.
- **Infeasible solutions :** Any point outside the feasible region is called infeasible solution.
- **Optimal (feasible) solution :** Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called optimal solution.

FUNDAMENTAL THEOREMS

- Let R be the feasible region (convex polygon) for a L.P.P. and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.
- Let R be the feasible region for a L.P.P. and let $Z = ax + by$ be the objective function.
  (i) If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point of R.
  (ii) If the feasible region R is unbounded, then a maximum or a minimum value of the objective function may or may not exist. However, if it exists, it must occur at a corner point of R.

GRAPHICAL METHODS FOR SOLVING L.P.P.

- **Corner Point Method :** For solving a L.P.P. the method comprises of the following steps :
  1. Find the feasible region of the L.P.P. and determine its corner points (vertices).
  2. Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m respectively be the largest and smallest values at these points.
  3. If the feasible region is bounded, M and m respectively are the maximum and minimum values of the objective function.
  4. If the feasible region is unbounded, then
  (i) M is the maximum value of the objective function, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, the objective function has no maximum value.
  (ii) m is the minimum value of the objective function, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.
- If two corner points of the feasible region are both optimal solutions of the same type, *i.e.,* both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.
- **Iso-Profit or Iso-Cost Method :** This is an alternate and more general method for finding the optimal solution of a L.P.P. In this method, we first give any suitable constant value, say Z_1 to the objective function and draw the corresponding line of the objective function. This line is called iso-profit or iso-cost line. Since every point on this line will yield the same profit or cost Z. After that we draw another line by giving another value say Z_2 to the objective function. The two lines are parallel to each other. If the objective function is of maximisation type, then we move the line corresponding to Z_1 to the line corresponding to Z_2 farthest from the origin and has atleast one point common to the feasible region. If the objective function is of minimisation type, then we move the line corresponding to Z_1 to the line corresponding to Z_2, nearest to the origin and has atleast one point common to the feasible region. The optimal solution will be obtained by the common point and the value of the objective function at these points gives the optimal solution.

DIFFERENT TYPES OF LINEAR PROGRAMMING PROBLEMS

- **Diet problems :** In diet problems, amount of different kinds of nutrients/constituents which are to be included in a diet are determined so

as to minimise the cost of the desired diet such that it contains a certain minimum amount of each nutrient/constituents.

- **Manufacturing problems (optimal product line problems) :** In such type of problems, the number of units of different products which is to be produced and sold by a firm when each product requires a fixed manpower, machine hour, labour hour per unit of product, warehouse space per unit of the output etc., are to be determined so as to achieve maximum profit.
- **Transportation problems :** In such type of problems, a transportation schedule, so as to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets is to be determined so as to minimise the cost of transportation subject to limitations (constraints) of demand of each market and supply from each plant or factory.

PROBABILITY

CONDITIONAL PROBABILITY

Let A and B be two events associated with a random experiment. Then the probability of occurrence of A under the condition that B has already occurred and $P(B) \neq 0$ is called conditional probability and it is denoted by $P(A|B)$.

$P(A|B)$ = Probability of occurrence of A given that B has already occurred.

$P(B|A)$ = Probability of occurrence of B given that A has already occurred.

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(A \cap B)}{P(B)}; P(B) \neq 0$$

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{P(A \cap B)}{P(A)}; P(A) \neq 0$$

Properties of Conditional Probability

- Let E and F be events of a sample space S of an experiment, then
  $P(S|F) = P(F|F) = 1$
- If A and B are any two events of a sample space S and F is an event of S such that $P(F) \neq 0$, then
  $P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$
  In particular, if A and B are disjoint events, then $P((A \cup B)|F) = P(A|F) + P(B|F)$
- $P(E'|F) = 1 - P(E|F)$

MULTIPLICATION THEOREM ON PROBABILITY

If A and B are two events associated with a random experiment, then

$$P(A \cap B) = P(A) \cdot P(B|A) ; P(A) \neq 0$$
$$P(A \cap B) = P(B) \cdot P(A|B) ; P(B) \neq 0$$

If $A_1, A_2, A_3, ..., A_n$ are n events related to a random experiment, then

$P(A_1 \cap A_2 \cap .... \cap A_n) = P(A_1) \cdot P(A_2|A_1) \times P(A_3|(A_1 \cap A_2)) \times .... \times P(A_n|(A_1 \cap A_2 \cap ... \cap A_{n-1}))$, where $P(A_n|(A_1 \cap A_2 \cap ..... \cap A_{n-1}))$ represents the conditional probability of the event A_n, given that the events $A_1, A_2, ..., A_{n-1}$ have happened.

INDEPENDENT EVENTS

Two events are said to be independent events iff the probability of occurrence or non-occurrence of one does not affect the probability of occurrence or non-occurrence of the other. A and B are two independent events if and only if

(i) $P(A|B) = P(A)$ (ii) $P(B|A) = P(B)$

(iii) $P(A \cap B) = P(A) \cdot P(B)$ or $P(A) \cdot P(B|A)$
(Multiplication theorem for independent events)

(iv) $P(A \cup B) = 1 - P(\overline{A}) \cdot P(\overline{B})$
(Addition theorem for independent events)

Note : (i) Two events A and B are said to be dependent if they are not independent, *i.e.* if

$P(A \cap B) \neq P(A) \cdot P(B)$

(ii) Three events A, B and C are said to be mutually independent if

$P(A \cap B) = P(A)\ P(B)$, $P(A \cap C) = P(A)\ P(C)$
$P(B \cap C) = P(B)\ P(C)$ and
$P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$

If atleast one of the above is not true for three given events, we say that the events are not independent.

TOTAL PROBABILITY LAW

If $\{E_1, E_2, ..., E_n\}$ be a set of n mutually exclusive and exhaustive events associated with a random experiment and A be any event associated with same experiment, then

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + ...$$
$$..... + P(E_n)P(A|E_n) = \sum_{i=1}^{n} P(E_i)P(A \mid E_i)$$

BAYES' THEOREM

If $E_1, E_2, E_3, ..., E_n$ be a set of n mutually exclusive and exhaustive events and A be any event associated with the same experiment, then

$$P(E_1 \mid A) = \frac{P(A \mid E_1)P(E_1)}{\{P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + ... \\ ..... + P(A \mid E_n)P(E_n)\}}$$

$$\Rightarrow \quad P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{\sum_{i=1}^{n} P(A \mid E_i)P(E_i)}$$

RANDOM VARIABLES

- A random variable is a real valued function X defined on the sample space S of a random experiment which associates a unique real number $X(x)$ to each $x \in S$.
  Domain of $X = S$, Range of $X = \{X(x) : x \in S\}$.
- If the random variable takes only a finite or an infinite but countable values, then it is called a discrete variable, and it is said to be continuous if it can assume any real value.
- **Probability Distribution of a Random Variable X :** Let X be a random variable defined on the sample space S of a random experiment, assume values $x_1, x_2, ..., x_n \in R$ with respective probabilities $p_1, p_2, ..., p_n$, then

X	:	x_1	x_2	$x_3 ..... x_n$
$P(X)$	:	p_1	p_2	$p_3 ..... p_n$

  is known as the probability distribution of X.
- **Mean of a discrete random variable**
  The mean of a random variable X is also called the expectation of X. *i.e.*,

$$E(X) = \mu = p_1x_1 + p_2x_2 + ... + p_nx_n \text{ or } \sum_{i=1}^{n} p_ix_i$$

- **Variance of a discrete random variable**

$$\text{Var}(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$$

OR

$$\text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i \right)^2 = E(X^2) - [E(X)]^2$$

$\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 p_i}$ is called the standard deviation of the random variable X.

BERNOULLI TRIALS

Trials of a random experiment are called Bernoulli trials, if they satisfies the following conditions:

(i) They should be finite in number.
(ii) They are independent of each other.
(iii) Each trial has exactly two outcomes : success or failure.
(iv) The probability of success or failure remains the same in each trial.

BINOMIAL DISTRIBUTION

- If we perform an experiment n times, then we shall obtain 2^n mutually exclusive and exhaustive outcomes. Define a random variable X on the sample space S as " the number of successes" in n trials, then X can take values 0, 1, 2, 3, ..., n. If r be any number from $\{0, 1, ..., n\}$, then $P(X = r)$ equals the probability of r successes $((n - r)$ failures) in any specified order multiplied by the number of times r success can occur in n trials.
  $P(X = r) = {}^nC_r p^r q^{n-r}$, where p is the probability of success and q is the probability of failure.
- **Mean and Variance of Binomial Distribution :**
  Mean = np and Variance = npq

VERY SHORT ANSWER TYPE (1 MARK)

1. The mean and variance of a binomial distribution are 12 and 3 respectively. Find the binomial distribution.
2. A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. State whether A and B are independent event or not.
3. If A and B are mutually exclusive events, find $P(A|B)$.
4. The corner points of the feasible region determined by the system of linear constraints are (0, 10), (5, 5), (15, 15), (0, 20). Let $Z = px + qy$, where $p, q > 0$. Find the condition on p and q so that the maximum of Z occurs at both the points (15, 15) and (0, 20).

5. The probability distribution of X is

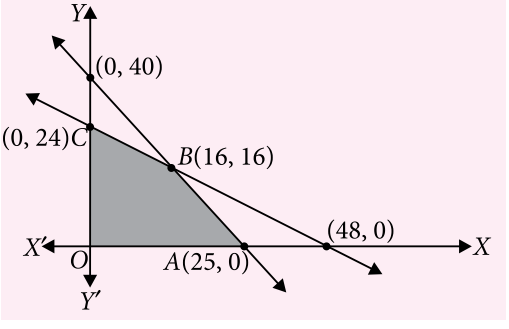
X	0	1	2	3
$P(X)$	0.2	k	k	$2k$

Write the value of k.

SHORT ANSWER TYPE (4 MARKS)

6. A student is given a test with 8 items of true-false type. If he gets 6 or more items correct, he is declared a pass. Given that he guesses the answer to each item, compute the probability that he will pass in the test.

7. Determine the maximum value of $Z = 4x + 3y$ if the feasible region for an L.P.P. is shown in figure.



8. The probability that student entering a university will graduate is 0.4. Find the probability that out of 3 students of the university :
(i) none will graduate,
(ii) only one will graduate,
(iii) all will graduate.

9. A family has two children. What is the probability that both the children are boys, given that atleast one of them is a boy?

LONG ANSWER TYPE (6 MARKS)

10. A manufacturer has three machines, I, II and III installed in his factory. Machines I and II are capable of being operated for atmost 12 hours whereas machine III must be operated for atleast 5 hours a day. He produces only two items M and N each requiring the use of all the three machines. The number of hours required for producing 1 unit each of M and N on the three machines are given in the following table :

Items	Number of hours required on Machine		
	I	II	III
M	1	2	1
N	2	1	1.25

He makes a profit of ₹ 600 and ₹ 400 on items M and N respectively. How many of each item should he produce so as to maximise his profit assuming that he can sell all the items that he produced? What will be the maximum profit?

11. A doctor is to visit a patient. From the past experience it is known that the probabilities of the doctor coming by train, bus, scooter or taxi are $\frac{1}{10}, \frac{1}{5}, \frac{3}{10}$ and $\frac{2}{5}$ respectively. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$ if he comes by train, bus or scooter respectively, but by taxi he will not be late. When he arrives, he is late. What is the probability that he comes by bus?

12. Find the probability distribution of the number of successes in two tosses of a dice, where success is defined as 'getting a number greater than 4'. Also, find the mean and variance of the distribution.

13. A man has ₹ 1,500 for purchasing rice and wheat. A bag of rice and a bag of wheat cost ₹ 180 and ₹ 120 respectively. He has the storage capacity of atmost 10 bags. He earns a profit of ₹ 11 and ₹ 9 per bag of rice and wheat respectively. Formulate the above problem as an L.P.P. to maximise the profit and solve it graphically.

14. Determine graphically the minimum value of the objective function

$Z = -50x + 20y$ ... (i)

Subject to the constraints :

$2x - y \geq -5$ ...(ii)

$3x + y \geq 3$ ...(iii)

$2x - 3y \leq 12$ ...(iv)

$x \geq 0, y \geq 0$ ... (v)

15. *A* speaks the truth 8 times out of 10 times. A die is tossed. He reports that it was 5. What is the probability that it was actually 5?

SOLUTIONS

1. Given $np = 12$; $npq = 3 \Rightarrow \frac{npq}{np} = \frac{3}{12} \Rightarrow q = \frac{1}{4}$

$\therefore \quad p = 1 - q = \frac{3}{4}$

As $np = 12 \Rightarrow n \times \frac{3}{4} = 12 \Rightarrow n = 16$

$\therefore$ Binomial distribution is $(q + p)^n = \left(\frac{1}{4} + \frac{3}{4}\right)^{16}$

2. We have, $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{6}$

Also, $P(A \cap B) = P$ (head appears on coin and 3 on the die) $= \frac{1}{12}$

Clearly, $P(A \cap B) = P(A) \times P(B)$

Thus, A and B are independent events.

3. Since A and B are mutually exclusive events, therefore $A \cap B = \phi$

Now, $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = 0$

4. Since maximum of Z occurs at (15, 15) and (0, 20) $\therefore$ $15p + 15q = 0 \cdot p + 20q \Rightarrow q = 3p$.

5. We have, $\Sigma P(X) = 1$

$\Rightarrow \quad 0.2 + 4k = 1 \Rightarrow \quad 4k = 0.8 \Rightarrow \quad k = 0.2$

6. Let

E_1 : Student makes a guess and gets true answer.

E_2 : Student makes a guess and gets false answer.

A : The event that the student will pass the test.

$\therefore \quad P(E_1) = P(E_2) = \frac{1}{2}$

$P(A \mid E_1) = \frac{3}{8}, \; P(A \mid E_2) = \frac{5}{8}$

$$\therefore \quad P(E_1 \mid A) = \frac{P(E_1) \cdot P(A \mid E_1)}{P(E_1) \cdot P(A \mid E_1) + P(E_2) \cdot P(A \mid E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{3}{8}}{\frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{5}{8}} = \frac{3}{8}$$

7. The feasible region is bounded. Therefore maximum of Z must occur at the corner points of the feasible region.

Corner Point	$Z = 4x + 3y$
$O(0, 0)$	$4(0) + 3(0) = 0$
$A(25, 0)$	$4(25) + 3(0) = 100$
$B(16, 16)$	$4(16) + 3(16) = 112 \leftarrow$ (Maximum)
$C(0, 24)$	$4(0) + 3(24) = 72$

Hence, the maximum value of Z is 112.

8. Let X denote the number of students who graduated.

Now, the probability that a student graduates, $p = 0.4$ $\therefore$ $q = 1 - p = 1 - 0.4 = 0.6$

(i) P(none will graduate) $= P(X = 0)$

$= {}^nC_0 p^0 q^{n-0} = {}^3C_0 (0.4)^0 (0.6)^3 = 0.216$

(ii) P (only one will graduate) $= P(X = 1)$

$= {}^nC_1 p^1 q^{n-1} = {}^3C_1 (0.4)^1 (0.6)^2 = 0.432$

(iii) P (all will graduate) $= P(X = 3)$

$= {}^nC_3 \cdot p^3 q^{n-3} = {}^3C_3 (0.4)^3 (0.6)^0 = 0.064$

9. Let B denote boy and G denote girl. Thus, we have

Sample Space $(S) = \{(B, G), (G, B), (B, B), (G, G)\}$

Now, let E be the event that both the children are boys and F be the event that atleast one of the child is a boy.

$\therefore \; E = \{(B, B)\} \Rightarrow \quad n(E) = 1$

$F = \{(B, G), (G, B), (B, B)\} \Rightarrow n(F) = 3$

$E \cap F = \{(B, B)\} \Rightarrow n(E \cap F) = 1$ and $n(S) = 4$

$\therefore \quad P(E) = \frac{1}{4}, P(F) = \frac{3}{4}, P(E \cap F) = \frac{1}{4}$

Thus, required probability $= P(E|F)$

$$= \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

10. Let x and y be the number of items of M and N produced by the three machines respectively.

Mathematical formulation of the given problem is as follows :

Maximise, $Z = 600x + 400y$

subject to the constraints :

$x + 2y \le 12$ ... (i)

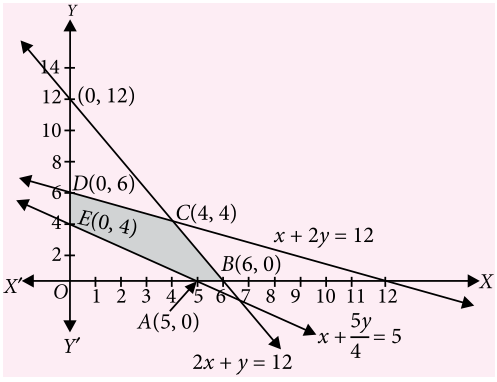
$2x + y \le 12$ ... (ii)

$x + \frac{5}{4} y \ge 5$ ... (iii)

$x, y \ge 0$ ...(iv)

Let us draw the graph of constraints (i) to (iv). $ABCDEA$ is the feasible region (shaded) as

shown in the figure. Observe that the feasible region is bounded, and coordinates of the corner points A, B, C, D and E are (5, 0), (6, 0), (4, 4), (0, 6) and (0, 4) respectively.



Let us evaluate $Z = 600x + 400y$ at these corner points.

Corner Point	$Z = 600x + 400y$	
$A(5, 0)$	3000	
$B(6, 0)$	3600	
$C(4, 4)$	4000	←(Maximum)
$D(0, 6)$	2400	
$E(0, 4)$	1600	

We clearly see that the point (4, 4) is giving the maximum value of Z. Hence, the manufacturer has to produce 4 units of each item to get the maximum profit of ₹ 4000.

11. Let A, B, C and D be the events that the doctor come by train, bus, scooter and taxi respectively.

$$\therefore \quad P(A)=\frac{1}{10},\ P(B)=\frac{1}{5},\ P(C)=\frac{3}{10} \text{ and}$$

$$P(D)=\frac{2}{5}$$

Let E be the event that doctor comes late.

Thus, $P(E\,|\,A)=\frac{1}{4}$, $P(E\,|\,B)=\frac{1}{3}$,

$P(E\,|\,C)=\frac{1}{12}$ and $P(E\,|\,D)=0$

$\therefore$ Probability that doctor comes late by bus,

$$P(B\,|\,E)=\frac{P(B)\cdot P(E\,|\,B)}{\{P(A)\cdot P(E\,|\,A)+P(B)\cdot P(E\,|\,B)+P(C)\cdot P(E\,|\,C)+P(D)\cdot P(E\,|\,D)\}}$$

$$=\frac{\frac{1}{5}\times\frac{1}{3}}{\left(\frac{1}{10}\times\frac{1}{4}\right)+\left(\frac{1}{5}\times\frac{1}{3}\right)+\left(\frac{3}{10}\times\frac{1}{12}\right)+\left(\frac{2}{5}\times 0\right)}$$

$$=\frac{\frac{1}{15}}{\frac{3+8+3}{120}}=\frac{4}{7}$$

12. Let X denote the number of successes in two tosses of a dice, then X can take values 0, 1, 2. Now, let S_i = Getting a success in i^{th} toss and , F_i = Getting a failure in i^{th} toss.

Then, $P(S_1)=\frac{2}{6}=\frac{1}{3}$

Also, $P(S_2)=\frac{1}{3}$

$\therefore \quad P(F_1)=P(F_2)=\frac{2}{3}$

Now, $P(X = 0)$ = Probability of getting no success in two tosses of a dice.

$$\Rightarrow \quad P(X=0)=P(F_1)\times P(F_2)=\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$$

$P(X = 1)$ = Probability of getting one success in two tosses of a dice.

$$P(X=1)=P(S_1)\cdot P(F_2)+P(F_1)\cdot P(S_2)$$
$$=\left(\frac{1}{3}\times\frac{2}{3}\right)+\left(\frac{2}{3}\times\frac{1}{3}\right)=\frac{4}{9}$$

$P(X = 2)$ = Probability of getting two successes in two tosses of a dice.

$$\Rightarrow \quad P(X=2)=P(S_1)\cdot P(S_2)=\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$$

Thus, the probability distribution of X is given by

X	0	1	2
$P(X)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Now,

$$\text{Mean} = \mu = \sum p_i x_i$$

$$=\left(0\times\frac{4}{9}\right)+\left(1\times\frac{4}{9}\right)+\left(2\times\frac{1}{9}\right)=\frac{6}{9}=\frac{2}{3}$$

and Variance, $\sigma^2=\sum p_i x_i^2-\left(\sum p_i x_i\right)^2$

$$=\left(\left(\frac{4}{9}\times 0\right)+\left(\frac{4}{9}\times 1\right)+\left(\frac{1}{9}\times 4\right)\right)-\left(\frac{2}{3}\right)^2=\frac{4}{9}$$

13. Let the number of rice bags be x and the number of wheat bags be y.

Maximise $Z = 11x + 9y$

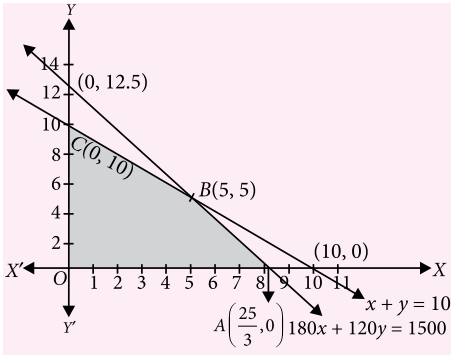
subject to the constraints,

$180x + 120y \le 1500$ ... (i)

$x + y \le 10$ ... (ii)

$x, y \ge 0$ ... (iii)

Now, plotting the graph of (i), (ii) and (iii), we get the required feasible region (shaded) as shown below. Also, feasible region is bounded.



We have corner points as

$A\left(\frac{25}{3}, 0\right)$, $B(5,5)$ and $C(0,10)$

Corner Points	$Z = 11x + 9y$
$A\left(\frac{25}{3}, 0\right)$	$\left(11 \times \frac{25}{3}\right) + (9 \times 0) = 91.67$
$B(5, 5)$	$(11 \times 5) + (9 \times 5) = 100$ (Maximum)
C(0, 10)	$(11 \times 0) + (9 \times 10) = 90$

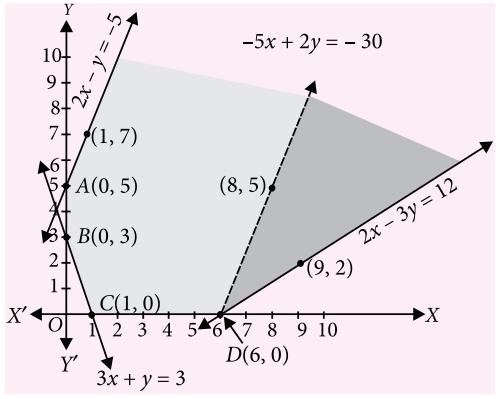
Therefore, $Z_{max} = ₹\ 100$

$\therefore$ Number of rice bags = 5 and number of wheat bags = 5

14. First of all, let us graph the feasible region of the system of inequalities (ii) to (v). The feasible region (shaded) is shown in the figure. Observe that the feasible region is unbounded.

Let us evaluate Z at the corner points.

Corner Point	$Z = -50x + 20y$	
$A(0, 5)$	100	
$B(0, 3)$	60	
$C(1, 0)$	–50	
$D(6, 0)$	–300	←(Minimum)



From this table, we find that – 300 is the smallest value of Z at the corner point $D(6, 0)$. As we know, if the region would have been bounded, then smallest value of Z is the minimum value of Z. But here we see that the feasible region is unbounded. Therefore, –300 may or may not be the minimum value of Z. To decide this, we graph the inequality,

$-50x + 20y < -300$ *i.e.*, $-5x + 2y < -30$

Since, open half plane has points in common with feasible region. Therefore, –300 will not be the minimum value of Z.

Therefore, $Z = -50x + 20y$ has no minimum value subject to the given constraints

15. Let E be the event that A reports that five occurs in the throwing of the dice and let S_1 be the event that five occurs and S_2 be the event that five does not occur. Then

$P(S_1)$ = Probability that five occurs = $\frac{1}{6}$

$P(S_2)$ = Probability that five does not occur = $\frac{5}{6}$

$P(E|S_1)$ = Probability that the man speaks the truth = $\frac{8}{10} = \frac{4}{5}$

$P(E|S_2)$ = Probability that the man does not speaks the truth = $1 - \frac{4}{5} = \frac{1}{5}$

Thus, by Bayes' theorem, we get

$$P(S_1|E) = \frac{P(S_1) \cdot P(E|S_1)}{P(S_1) \cdot P(E|S_1) + P(S_2) \cdot P(E|S_2)}$$

$$= \frac{\frac{1}{6} \times \frac{4}{5}}{\frac{1}{6} \times \frac{4}{5} + \frac{5}{6} \times \frac{1}{5}} = \frac{4}{4+5} = \frac{4}{9}$$

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. The line with direction cosines proportional to 2, 1, 2 meets each of the lines $x = y + 1 = z$ and $x + 1 = 2y = 2z$. The coordinates of each of the points of intersection are given by

(a) (3, 3, 3), (1, 1, 1) (b) (3, 2, 3), (1, 1, 1)

(c) (3, 2, 3), (1, 1, 2) (d) (2, 3, 3), (2, 1, 1)

2. Total number of common tangents of $y^2 = 4ax$ and $xy = c^2$, is equal to

(a) 1 (b) 2

(c) 3 (d) 4

3. Let $f : R \to R$ be a function defined by $f(x) = [x]^2 + [x + 1] - 3$ {where $[\cdot]$ denotes greatest integer function}, then $f(x)$ is

(a) many–one into (b) many–one onto

(c) one–one into (d) one–one onto

4. If $2^{3n} - \alpha n - \beta$ is divisible by 49, then (α, β) is, $n \in N$

(a) (–7, –1) (b) (7, 1)

(c) (49, 1) (d) (49, 7)

5. The length of projection of the line segment joining the points (1, –1, 0) and (–1, 0, 1) to the plane $2x + y + 6z = 1$ is equal to

(a) $\sqrt{\frac{255}{41}}$ units (b) $\sqrt{\frac{237}{41}}$ units

(c) $\sqrt{\frac{137}{41}}$ units (d) $\sqrt{\frac{155}{41}}$ units

6. $\sum_{r=1}^{15} \frac{r2^r}{(r+2)!}$ is equal to

(a) $\frac{17! - 2^{16}}{17!}$ (b) $\frac{18! - 2^{17}}{18!}$

(c) $\frac{16! - 2^{15}}{16!}$ (d) $\frac{15! - 2^{14}}{15!}$

7. There are n locks and n matching keys. If all the locks and keys are to be perfectly matched, then maximum number of trials is equal to

(a) ${}^nC_2$ (b) ${}^{n-1}C_2$

(c) ${}^{n+1}C_2$ (d) none of these

8. Maximum length of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that eccentric angles of its extremities differ by $\frac{\pi}{2}$, $(a > b)$ is

(a) $a\sqrt{2}$ (b) $b\sqrt{2}$

(c) $ab\sqrt{2}$ (d) none of these

9. The number of ordered pairs (m, n) $(m, n \in \{1, 2, \ldots, 20\})$ such that $3^m + 7^n$ is a multiple of 10, is

(a) 100 (b) 200

(c) $4! \times 4!$ (d) none of these

10. If the length of the tangent drawn from a variable point to one given circle is $k(\neq 1)$ times the length of the tangent from it to another circle, then the locus of the variable point is a/an

(a) ellipse (b) parabola

(c) circle (d) hyperbola

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SOLUTIONS

1. (b) : The first line $x = y + 1 = z = \mu$ gives general point as $(\mu, \mu - 1, \mu)$. Second line $x + 1 = 2y = 2z = \lambda$ gives general point as $(\lambda-1, \lambda/2, \lambda/2)$. The ratios of direction ratios of line joining points of intersection to direction cosines proportional to 2, 1, 2 are same.

$$\Rightarrow \quad \frac{\mu-\lambda+1}{2}=\frac{\mu-1-\frac{\lambda}{2}}{1}=\frac{\mu-\frac{\lambda}{2}}{2}$$

$$\Rightarrow \quad \mu-\lambda+1=2\mu-2-\lambda=\mu-\frac{\lambda}{2}$$

Solving, we get $\mu = 3$, $\lambda = 2$

$\therefore$ Coordinates of point of intersection are (3, 2, 3), (1, 1, 1)

2. (a) : Any tangent to $xy = c^2$ is $\frac{x}{t}+yt=2c$

$$i.e.\ y=-\frac{x}{t^2}+\frac{2c}{t}$$

Comparing it with $y=mx+\frac{a}{m}$, we get

$$\frac{1}{1}=\frac{mt^2}{-1}=\frac{at}{2cm} \Rightarrow m=-\frac{1}{t^2},\ m=\frac{at}{2c}$$

$$\Rightarrow \quad \frac{at}{2c}=\frac{-1}{t^2} \Rightarrow \quad t=-\left(\frac{2c}{a}\right)^{1/3}$$

So, there is only one common tangent.

3. (a) : $f(x) = [x]^2 + [x] - 2$

$= ([x] + 2)([x] - 1)$

Now, $f(x) = 0$ for $x \in [1, 2)$ and $x \in [-2, -1)$

So $f(x)$ is many-one. As $f(x)$ will take only integral values, so it is into.

4. (b) : $8^n - \alpha n - \beta = (1 + 7)^n - \alpha n - \beta$

$= 1 + 7n + {}^nC_2 7^2 + {}^nC_3 7^3 + ... + {}^nC_n 7^n - \alpha n - \beta$

$= (1 - \beta) + (7 - \alpha)n + 49({}^nC_2 + {}^nC_3 7 + ...)$

$\therefore$ If $8^n - \alpha n - \beta$ is divisible by 49, then

$1 - \beta = 0$, $7 - \alpha = 0$

$\Rightarrow \quad \beta = 1, \alpha = 7.$

5. (b) : $A \equiv (1, -1, 0)$, $B \equiv (-1, 0, 1)$

Direction ratios of segment AB are $(2, -1, -1)$.

If θ be the acute angle between segment AB and normal to plane

$$\cos\theta=\frac{|(2\times2)-(1\times1)-(1\times6)|}{\sqrt{4+1+36}\sqrt{4+1+1}}=\frac{3}{\sqrt{246}}$$

$\therefore$ Length of projection

$$= AB\sin\theta=\sqrt{6}\sqrt{1-\frac{9}{246}}=\sqrt{\frac{237}{41}}\ \text{units}$$

6. (a) : $\frac{r2^r}{(r+2)!}=\frac{(r+2-2)2^r}{(r+2)!}$

$$=\frac{2^r}{(r+1)!}-\frac{2^{r+1}}{(r+2)!}=-\left(\frac{2^{r+1}}{(r+2)!}-\frac{2^r}{(r+1)!}\right)$$

$= (V_r - V_{r+1})$

$$\Rightarrow \quad \sum_{r=1}^{15}\frac{r2^r}{(r+2)!}=-\left(\frac{2^{16}}{17!}-\frac{2}{2!}\right)=1-\frac{2^{16}}{17!}$$

7. (c) : For the first key, maximum number of trials needed is n. For second key, it will be $(n - 1)$.

In general, for r^{th} key, maximum number of trials needed is $(n - r + 1)$.

Thus total number of trials needed

$$=n+(n-1)+(n-2)+....+1=\frac{n(n+1)}{2}={}^{n+1}C_2\,.$$

8. (a) : Let the extremities of the chord be $P_1 \equiv (a\cos\theta, b\sin\theta)$ and $P_2 \equiv (-a\sin\theta, b\cos\theta)$

$$\text{Now, } (P_1P_2)^2=a^2(\cos\theta+\sin\theta)^2+b^2(\sin\theta-\cos\theta)^2$$

$$\Rightarrow \quad (P_1P_2)^2=a^2+b^2+(a^2-b^2)\sin2\theta$$

$$\leq a^2+b^2+a^2-b^2$$

$$\Rightarrow \quad (P_1P_2)^2\leq 2a^2 \Rightarrow \quad P_1P_2\leq a\sqrt{2}$$

9. (a) : The last digit of any power of 3 can be 3, 9, 7, 1. Similarly last digit of any power of 7 can be 7, 9, 3, 1.

$\Rightarrow$ Total number of ways = $(5 \times 5) + (5 \times 5) + (5 \times 5) + (5 \times 5) = 100$.

10. (c) : $\sqrt{S_1}=k\sqrt{S_2}$

$$\Rightarrow \quad S_1=k^2S_2$$

So locus is $x^2 + y^2 + 2g_1x + 2f_1y + c_1$

$= k^2(x^2 + y^2 + 2g_2x + 2f_2y + c_2)$

$\Rightarrow \quad (x^2 + y^2)(1 - k^2) + 2x(g_1 - k^2g_2)$

$+ 2y(f_1 - k^2f_2) + c_1 - k^2c_2 = 0$

which is a circle.

■■

MOCK TEST PAPER 2015

JEE Main & Advanced

Series-7

Co-ordinate Geometry of Two Dimensions : Definition of conic section, Directrix, Focus and Eccentricity, classification based on eccentricity.

Co-ordinate Geometry of Three Dimensions : Direction cosines and direction ratios, distance between two points and section formula, equation of a straight line, equation of a plane, distance of a point from a plane.

Application of Calculus : Tangents and normals, conditions of tangency. Determination of monotonicity, maxima and minima.

Vectors : Addition of vectors, scalar multiplication, dot and cross products, scalar triple product.

PAPER-I

SECTION-I

Single Correct Option

This section contains 10 multiple choice questions. Each question has 4 choices, out of which ONLY ONE is correct.

1. If f is an increasing function and g is a decreasing function such that $g(f(x))$ exist, then

(a) $g(f(x))$ is an increasing function.
(b) $g(f(x))$ is a decreasing function.
(c) nothing can be said.
(d) $g(f(x))$ is a constant function.

2. The number of tangents to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ which are equally inclined to the axes, is

(a) 4 (b) 3 (c) 2 (d) 1

3. The equation of one of the tangents to the curve $y = \cos(x + y)$, the normal at which is parallel to the line $2x - y = 0$ in $-2\pi < x < 2\pi$ is

(a) $x + 2y = 1$ (b) $x + 2y = \frac{\pi}{2}$
(c) $x + 2y = \frac{\pi}{4}$ (d) $x + 2y = \frac{\pi}{3}$

4. In $(0, 2\pi)$, the expression $\sin\theta + \cos^2\theta$ has

(a) one point of maximum and one point of minimum.
(b) the highest value $= \frac{5}{4}$.
(c) only one point of maximum.
(d) neither a point of maximum nor a point of minimum.

5. A particle moves in a straight line with a velocity given by $\frac{dx}{dt} = x + 1$ (x is the distance travelled). The time taken by a particle to travel a distance of 99 meters is

(a) $\log_{10} e$ (b) $2 \log_e 10$
(c) $2 \log_{10} e$ (d) $1/2 \log_{10} e$

6. For all x in $[0, 1]$, let the second derivative $f''(x)$ of a function exist and satisfy $|f''(x)| \le 1$. If $f(0) = f(1)$, then in $(0, 1)$

(a) $|f(x)| > 1$ (b) $|f(x)| < 1$
(c) $|f'(x)| > 1$ (d) $|f'(x)| < 1$

7. $f(x) = -1 + kx + k$ neither touches nor cuts the curve $f(x) = \ln x$, then minimum value of $k \in$

(a) $\left(\frac{1}{e}, \frac{1}{\sqrt{e}}\right)$ (b) (e, e^2)
(c) $\left(\frac{1}{\sqrt{e}}, e\right)$ (d) none of these

8. If $\vec{u} = \vec{a} - \vec{b}, \vec{v} = \vec{a} + \vec{b}$ and $|\vec{a}| = |\vec{b}| = 2$, then $|\vec{u} \times \vec{v}|$ is

(a) $2\sqrt{16 - (\vec{a} \cdot \vec{b})^2}$ (b) $2\sqrt{4 - (\vec{a} \cdot \vec{b})^2}$
(c) $\sqrt{16 - (\vec{a} \cdot \vec{b})^2}$ (d) $\sqrt{4 - (\vec{a} \cdot \vec{b})^2}$

9. A straight line $\vec{r} = \vec{a} + \lambda\vec{b}$ meets the plane $\vec{r} \cdot \vec{n} = 0$ at P. The position vector of P is

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(a) $\vec{a}+\left(\frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\right)\vec{b}$ (b) $\vec{b}-\left(\frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\right)\vec{a}$

(c) $\vec{a}-\left(\frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\right)\vec{b}$ (d) none of these

10. The point at which the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ meets the plane $x-2y+z=20$ is

(a) (7, –8, 26) (b) (8, 7, 26)

(c) (7, 8, 26) (d) none of these

SECTION-II

Multiple Correct Option

This section contains 5 multiple choice questions. Each question has 4 choices, out of which ONE or MORE is/are correct.

11. Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is of magnitude $\sqrt{\frac{2}{3}}$ is

(a) $2\hat{i}+3\hat{j}-3\hat{k}$ (b) $2\hat{i}+3\hat{j}+3\hat{k}$

(c) $-2\hat{i}-\hat{j}+5\hat{k}$ (d) $2\hat{i}+\hat{j}+5\hat{k}$

12. Let $\vec{a}$ and $\vec{b}$ are two non-collinear unit vectors. If $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$, then $|\vec{v}|$ is

(a) $|\vec{u}|$ (b) $|\vec{u}|+|\vec{u}\cdot\vec{a}|$

(c) $|\vec{u}|+|\vec{u}\cdot\vec{b}|$ (d) $|\vec{u}|+\vec{u}\cdot(\vec{a}+\vec{b})$

13. The lines $\frac{x-4}{1}=\frac{y-3}{1}=\frac{6-z}{k}$ and $\frac{x-3}{k}=\frac{y-4}{2}=\frac{z-7}{1}$ are coplanar if k equals,

(a) 0 or –3 (b) –1 or –3

(c) 2 or –3 (d) –3 or –2

14. The plane $7x-2y+z+21=0$

(a) contains the point (–1, 7, 0).

(b) contains the line $\frac{x+2}{1}=\frac{3-y}{-2}=\frac{z+1}{-3}$.

(c) is perpendicular to the line $\frac{x}{7}=\frac{y}{-2}=\frac{z}{1}$.

(d) is parallel to the plane $7x-2y+z+5=0$.

15. The tangent PT and normal PN to the parabola $y^2=4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of ΔPTN is a parabola whose

(a) vertex is $\left(\frac{2a}{3},0\right)$ (b) directrix is $x=0$

(c) latus rectum is $\frac{2a}{3}$ (d) focus is $(a, 0)$

SECTION-III

Assertion and Reason Type

In each of the following questions two statements are given. Statement-1 (Assertion) and Statement-2 (Reason). Examine the statements carefully and answer the questions according to the instructions given below.

(a) If both Statement-1 and Statement-2 are correct and Statement-2 is the proper explanation of Statement-1.

(b) If both Statement-1 and Statement-2 are correct and Statement-2 is not the proper explanation of Statement-1.

(c) If Statement-1 is correct and Statement-2 is wrong.

(d) If Statement-1 is wrong and Statement-2 is correct.

16. Statement-1 : If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ is vertical, the line of greatest slope on a plane with normal $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ is along $\hat{i}-4\hat{j}+2\hat{k}$.

Statement-2 : If $\vec{a}$ is vertical, then the line of greatest slope on a plane with normal $\vec{b}$ is along $(\vec{a}\times\vec{b})\times\vec{b}$.

17. Statement-1 : If Rolle's theorem is applicable for function $f(x)$, then $f(x)$ is many-one function.

Statement-2 : L.M.V. theorem is applicable for $f(x)$, then $f(x)$ is one-one.

18. Statement-1 : If $f(x) = (x-5)^3$, $f(x)$ has neither maximum nor minimum at $x = 5$.

Statement-2 : $f'(x) = 0$, $f''(x) = 0$ when $x = 5$.

19. Statement-1 : A tangent parallel to x-axis can be drawn for $f(x) = (x-2)(x-3)(x-4) \forall x \in [2, 4]$.

Statement-2 : A horizontal tangent can be drawn in case of Rolle's theorem.

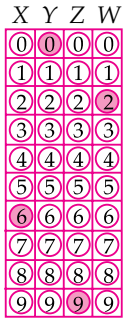
20. Statement-1 : Angle between the planes $2x-y+z=6$ and $x+y+2z=7$ is 90°.

Statement-2 : Angle between the two planes is same as the angle between their normals *i.e.*, if θ is the angle between the normals $\vec{n}_1$ and $\vec{n}_2$, then $\cos\theta=\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$.

SECTION-IV

Integer Answer Type

This section contains 5 questions. The answer to each of the questions is a single-digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like as given.



21. Let $P(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x\to 0}\left(1+\frac{P(x)}{x^2}\right)=2$. Then the value of $P(2)$ is

22. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is

23. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

is $\sqrt{6}$, then $|d|$ is

24. If $\vec{a}$ and $\vec{b}$ are two vectors in space given by $\vec{a}=\frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b}=\frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then the value of $(2\vec{a}+\vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})]$ is

25. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is

PAPER-II

SECTION-I

Single Correct Option

This section contains 10 multiple choice questions. Each question has 4 choices, out of which ONLY ONE is correct.

1. Which of the following function satisfies the conditions of the Rolle's theorem on the interval $[-1, 1]$?

(a) $f(x) = |x|[x]$ (b) $f(x) = [x] + [-x]$

(c) $f(x)=\frac{\sin x}{x}$ for $x\neq 0,\ f(0)=0$

(d) $f(x) = |x| - |\sin x|$

2. The direction cosine of a line which are connected by the relation $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$ are

(a) $\left(\frac{-1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$ (b) $\left(\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{-1}{\sqrt{6}}\right)$

(c) $\left(\frac{1}{\sqrt{4}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$ (d) $\left(\frac{1}{\sqrt{4}},\frac{-2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$

3. If $\overrightarrow{OA}=\hat{i}+3\hat{j}-2\hat{k}$ and $\overrightarrow{OB}=3\hat{i}+\hat{j}-2\hat{k}$, then $\overrightarrow{OC}$ which bisects the angle AOB is given by

(a) $\hat{i}-\hat{j}-\hat{k}$ (b) $\hat{i}+\hat{j}+\hat{k}$

(c) $-\hat{i}+\hat{j}-\hat{k}$ (d) $\hat{i}+\hat{j}-\hat{k}$

4. The distance of the point (3, 0, 5) from the line $x - 2y + 2z - 4 = 0 = x + 3z - 11$ is

(a) 1 (b) 2 (c) 3 (d) 4

5. The shortest distance between the lines

$\vec{r}=s(2\hat{i}+3\hat{j}+4\hat{k})-(\hat{i}+\hat{j}+\hat{k})$

and $\vec{r}=t(3\hat{i}+4\hat{j}+5\hat{k})-\hat{i}$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\frac{1}{\sqrt{6}}$

6. Let $f(x)$, a polynomial of degree 3 has maximum at $x = -1$. If $f'(x)$ has minimum at $x = 1$, then $f(x)$ has minimum at $x =$

(a) 0 (b) 1 (c) 2 (d) 3

7. The slope of the straight line which is both tangent and normal to the curve $4x^3 = 27y^2$ is

(a) ± 1 (b) $\pm\frac{1}{2}$ (c) $\pm\frac{1}{\sqrt{2}}$ (d) $\pm\sqrt{2}$

8. Let $g(x)=2f\left(\frac{x}{2}\right)+f(2-x)$ and $f''(x) < 0$ $\forall\ x \in (0, 2)$. Then $g(x)$

(a) increases in (0, 1) and decreases in (1, 2).

(b) decreases in (0, 1) and increases in (1, 2).

(c) increases in $\left(0,\frac{4}{3}\right)$ and decreases in $\left(\frac{4}{3},2\right)$.

(d) decreases in $\left(0,\frac{4}{3}\right)$ and increases in $\left(\frac{4}{3},2\right)$.

9. The three sides of a trapezium are each equal to 2. The maximum area of the trapezium is

(a) $5\sqrt{3}$ (b) $4\sqrt{3}$ (c) $3\sqrt{3}$ (d) $2\sqrt{3}$

10. A ray of light emancipating from the point source $P(1, -3, 2)$ and travelling parallel to the line $\frac{x-2}{1}=\frac{y}{3}=\frac{z+1}{2}$ is incident on the plane $x+y-3z=0$ at the point Q. After reflecting from the plane the ray travels along the line QR. It is also known that the incident ray, reflected ray and the normal to the plane at the point of incidence are in the same plane. The point Q is

(a) (3, 15, 6) (b) (3, 6, 3)
(c) (–3, –6, –3) (d) (–3, –15, –6)

SECTION-II

Multiple Correct Option

This section contains 5 multiple choice questions. Each question has 4 choices, out of which ONE or MORE is/are correct.

11. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of the removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are

(a) 24 (b) 32 (c) 45 (d) 60

12. The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius r, intersects the first circle at R above the x-axis and the line segment PQ at S.

(a) Coordinate of R is $\left(\frac{2-r^2}{2}, \frac{r\sqrt{4-r^2}}{2}\right)$.

(b) Area of $\Delta QRS=\frac{1}{2}r\frac{r\sqrt{4-r^2}}{2}$.

(c) ΔQRS is maximum, when $r=\frac{2\sqrt{2}}{3}$.

(d) Maximum area of $\Delta QRS=\frac{4\sqrt{3}}{9}$.

13. $f(x)$ is a cubic polynomial with $f(2) = 18$ and $f(1) = -1$. Also, $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minima at $x = 0$, then

(a) the distance between $(-1, 2)$ and $(a, f(a))$ where $x = a$ is the point of local minima is $2\sqrt{5}$.

(b) $f(x)$ is increasing for $x\in[1, 2\sqrt{5}]$.

(c) $f(x)$ has local minima at $x = 1$.

(d) the value of $f(0) = 5$.

14. Let $y = f(x)$ be the equation of a parabola which is touched by the line $y = x$ at the point where $x = 1$. Then

(a) $f'(0) = f'(1)$ (b) $f'(1) = 1$
(c) $f(0) + f'(0) + f''(0) = 1$
(d) $2f(0) = 1 - f'(0)$

15. Let $f(x) = (x - 1)^4(x - 2)^n$, $n \in N$. Then $f(x)$ has

(a) a maximum at $x = 1$, if n is odd.
(b) a maximum at $x = 1$, if n is even.
(c) a minimum at $x = 2$, if n is even.
(d) a maximum at $x = 2$, if n is odd.

SECTION-III

Linked Comprehension Type

This section contains 4 paragraphs. Based upon each paragraph, 2 multiple choice questions have to be answered. Each question has 4 choices, out of which ONE or MORE is/are correct.

Paragraph for Question 16 and 17

Let L_1 be the line $2x-2y+3z-2=0=x-y+z+1$ and L_2 be the line $x + 2y - z - 3 = 3x - y + 2z - 1 = 0$.

16. The distance of the origin from the plane through L_1 and L_2 is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{\sqrt{15}}$

17. L_1 and L_2 meet at a point P whose distance from the origin is

(a) $\sqrt{33}$ (b) $\sqrt{22}$ (c) $\sqrt{25}$ (d) $2\sqrt{41}$

Paragraph for Question 18 and 19

Let $f : [0, 1] \to R$ (the set of real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$.

18. Which of the following is true for $0 < x < 1$?

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2}<f(x)<\frac{1}{2}$
(c) $-\frac{1}{4}<x<1$ (d) $-\infty < f(x) < 0$

19. If the function $e^{-x}f(x)$ assumes its minimum in the interval [0, 1] at $x=\frac{1}{4}$, which of the following is true?

(a) $f'(x)<f(x), \frac{1}{4}<x<\frac{3}{4}$

(b) $f'(x)>f(x), 0<x<\frac{1}{4}$

(c) $f'(x)<f(x), 0<x<\frac{1}{4}$

(d) $f'(x)<f(x), \frac{3}{4}<x<1$

Paragraph for Question 20 and 21

Perpendiculars AP, AQ and AR are drawn to x, y and z-axes, respectively, from the point $A(1, -1, 2)$.

20. The A.M. of AP^2, AQ^2 and AR^2 is

(a) 4 (b) 5 (c) 3 (d) 2

21. The equation of the plane PQR is

(a) $2x + 2y - z = 2$ (b) $2x - 2y + z = 2$

(c) $2x - y + z = 2$ (d) $x - 2y + z = 2$

Paragraph for Question 22 and 23

Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors where $\vec{\alpha}=\vec{b}\times\vec{c}$, $\vec{\beta}=\vec{c}\times\vec{a}$ and $\vec{\gamma}=\vec{a}\times\vec{b}$.

22. $(\vec{\alpha}\times\vec{\beta})\cdot\vec{\gamma}=$

(a) $[\vec{a}\,\vec{b}\,\vec{c}]$ (b) $[\vec{a}\,\vec{b}\,\vec{c}]^2$ (c) $[\vec{a}\,\vec{b}\,\vec{c}]^4$ (d) 0

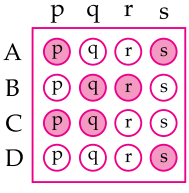
23. $[\vec{\beta}\times\vec{\gamma}\ \ \vec{\gamma}\times\vec{\alpha}\ \ \vec{\alpha}\times\vec{\beta}]=$

(a) $[\vec{a}\,\vec{b}\,\vec{c}]$ (b) $[\vec{a}\,\vec{b}\,\vec{c}]^2$ (c) $[\vec{a}\,\vec{b}\,\vec{c}]^4$ (d) 0

SECTION-IV

Matrix Match Type

This section contains 2 questions. Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled as A, B, C and D, while the statements in Column-II are labelled as p, q, r and s. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example. If the correct matches are A-p and s ; B-q and r; C-p and q; and D-s, then the correct darkening of bubbles will look like the above.



24. Match the following:

Column I		Column II	
(A)	Volume of parallelopiped determined by vectors $\vec{a}, \vec{b}\,\vec{c}$ is 2. Then, the volume of the parallelopiped determined by vectors $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	(p)	100
(B)	Volume of parallelopiped determined by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 5. Then, the volume of the parallelopiped determined by vectors $3(\vec{a}+\vec{b}), (\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	(q)	30
(C)	Area of the triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then, the area of the triangle with adjacent sides determined by vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	(r)	24
(D)	Area of the parallelogram with adjacent sides determined by the vectors $\vec{a}$ and $\vec{b}$ is 30. Then, the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	(s)	60

25. Match the following:

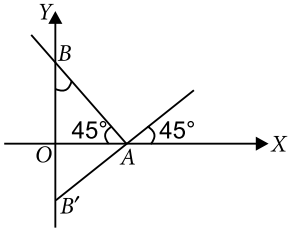
Column I		Column II	
(A)	The plane containing the points (1, 1, 2), (1, 2, 1), (2, 1, 1) and (1, –1, 4) is	(p)	$x - 5y + 3z = 0$
(B)	The plane through origin and containing the line $\frac{x-1}{5}=\frac{y-2}{4}=\frac{z-3}{5}$ is	(q)	$9x - 3y - z - 22 = 0$
(C)	The plane containing the line $\vec{r}=3\hat{i}+5\hat{k}+s(\hat{i}+2\hat{j}+3\hat{k})$ is at right angle to the plane $2x + 7y - 3z = 0$ is	(r)	$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=4$
(D)	The plane containing the point (3, –4, 5) and the line of intersection of the planes $2x - 3y + z - 11 = 0$ and $x + 4y - 2z + 6 = 0$ is	(s)	$4x + 5y - 3z + 23 = 0$

SOLUTIONS

PAPER-I

1. (b) : Since $f(x)$ is an increasing function,
Therefore $f'(x) > 0$ and $g(x)$ is a decreasing function
$\therefore \quad g'(x) < 0$
Now $g(f(x))' = g'(f(x)) \cdot f'(x) < 0$
$\therefore \quad g(f(x))$ is a decreasing function.

2. (a) : Given that, the tangent is equally inclined to the axes.
$\therefore$ It must be either AB or $B'A$ as shown in the figure.



$\therefore$ Slope of the tangents are is ± 1, *i.e.*, $\frac{dy}{dx} = \pm 1$.

Now, the given curve is $x^{2/3} + y^{2/3} = a^{2/3}$

$$\therefore \quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3} \Rightarrow \quad -\left(\frac{y}{x}\right)^{1/3} = \pm 1$$

$$\Rightarrow \quad y = \pm x = \pm \frac{a}{2\sqrt{2}}$$

$\therefore$ There are four points of contact, one each in 4 quadrants.

3. (b) : We have, $y = \cos(x + y)$

$$\Rightarrow \quad \frac{dy}{dx} = -\sin(x+y)\left(1 + \frac{dy}{dx}\right)$$

The normal is parallel to $2x - y = 0$
$\Rightarrow$ The tangent is parallel to $x + 2y = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}.$$

$$\text{So,} \quad -\sin(x+y)\left(1 - \frac{1}{2}\right) = -\frac{1}{2} \Rightarrow \frac{1}{2}\sin(x+y) = \frac{1}{2}$$

$$\Rightarrow \quad \sin(x+y) = 1 \Rightarrow \cos(x+y) = 0$$

$$\Rightarrow \quad x + y = \frac{\pi}{2} \quad \text{or} \quad -\frac{3\pi}{2}$$

$\therefore$ The point of contact is $\left(\frac{\pi}{2}, 0\right)$ or $\left(-\frac{3\pi}{2}, 0\right)$.

At $\left(\frac{\pi}{2}, 0\right)$, the equation of tangent is $x + 2y = \frac{\pi}{2}$.

4. (b) : $\sin\theta + \cos^2\theta = \sin\theta + 1 - \sin^2\theta$
$= 1 - (\sin^2\theta - \sin\theta)$

$$= 1 - \left\{(\sin\theta)^2 - 2\sin\theta \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}$$

$$= 1 - \left(\sin\theta - \frac{1}{2}\right)^2 + \frac{1}{4} = \frac{5}{4} - \left(\sin\theta - \frac{1}{2}\right)^2$$

Thus, the given expression $\sin\theta + \cos^2\theta$ will be maximum when $\left(\sin\theta - \frac{1}{2}\right)^2 = 0$

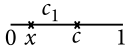
$$\Rightarrow \quad \sin\theta = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6}$$

$$\text{Also } \sin\theta = \frac{1}{2} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{5\pi}{6} \quad \therefore \quad \theta = \frac{5\pi}{6}$$

Thus the given function has two points of maximum, namely $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

5. (b) : We have, $\frac{dx}{dt} = x + 1 \Rightarrow \log_e(x+1) = t + c$
Putting $t = 0$, $x = 0$, we get $c = 0$
$\therefore \quad t = \log_e(x + 1) \Rightarrow t = \log_e 100$ [Putting $x = 99$]
$\Rightarrow \quad t = 2\log_e 10$

6. (d) : $f(0) = f(1) \Rightarrow f'(c) = 0,\ 0 < c < 1$ (by Rolle's theorem)



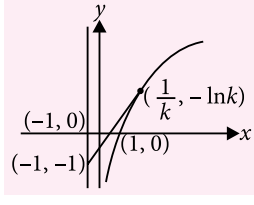
Let x be any point in (0, 1) other than c.
By Lagrange's theorem,
$f'(x) - f'(c) = (x - c)\,|f''(c_1)|$
$\Rightarrow \quad |f'(x)| = |x - c|\,|f''(c_1)| < 1 \quad \because \ |x - c| < 1$ and
$\Rightarrow \quad |f'(x)| < 1\ |f''(c_1)| \le 1$

7. (a) : We have, $f(x) = kx + k - 1$ ...(i)
Clearly the above line always passes through (–1, –1).
Maximum slope can go upto ∞. For minimum slope this line should touch $y = \ln x$
For minimum slope,

$$\frac{dy}{dx} = \frac{1}{x} = k \quad \Rightarrow \quad x = \frac{1}{k}$$

$$\therefore \quad \left(\frac{1}{k}, \ln\frac{1}{k}\right) \textit{ i.e.,} \left(\frac{1}{k}, -\ln k\right)$$

is the point of tangency.



$$\therefore \quad \frac{-\ln k + 1}{\frac{1}{k} + 1} = k \Rightarrow -\ln k + 1 = 1 + k \Rightarrow -\ln k = k$$

Let $f(k) = k + \ln k$

$$f\left(\frac{1}{e}\right) = \frac{1}{e} - 1 < 0,\ f\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{\sqrt{e}} - \frac{1}{2} > 0$$

So, one root must lie between $\frac{1}{e}$ and $\frac{1}{\sqrt{e}}$.

8. (a) : Given, $\vec{u} = \vec{a} - \vec{b}, \vec{v} = \vec{a} + \vec{b}$ and $|\vec{a}| = |\vec{b}| = 2$

$\vec{u} \times \vec{v} = (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$= (\vec{a} \times \vec{a}) + (\vec{a} \times \vec{b}) - (\vec{b} \times \vec{a}) - (\vec{b} \times \vec{b})$

$\Rightarrow \vec{u} \times \vec{v} = 2(\vec{a} \times \vec{b})$ [$\because \vec{a} \times \vec{a} = 0$ and $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$]

$\therefore |\vec{u} \times \vec{v}| = 2|\vec{a}||\vec{b}| \sin\theta$, (where θ is the angle between $\vec{a}$ and $\vec{b}$)

$$= 8\sin\theta = 8\sqrt{1 - \cos^2\theta} = 8\sqrt{1 - \left\{\frac{\vec{a} \cdot \vec{b}}{4}\right\}^2}$$

$$= 2\sqrt{16 - (\vec{a} \cdot \vec{b})^2}$$

9. (c) : The straight line $\vec{r} = \vec{a} + \lambda\vec{b}$ meets the plane $\vec{r} \cdot \vec{n} = 0$ at P for which λ is given by

$$(\vec{a} + \lambda\vec{b}) \cdot \vec{n} = 0 \Rightarrow \vec{a} \cdot \vec{n} + \lambda\vec{b} \cdot \vec{n} = 0$$

$$\Rightarrow \lambda = -\frac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}}$$

Thus, the position vector of P is $\vec{r} = \vec{a} - \frac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}}\vec{b}$

10. (b) : Any point on the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ is $(3r + 2, 4r - 1, 12r + 2)$.

Since the point lies in the plane $x - 2y + z = 20$

$\therefore (3r + 2) - 2(4r - 1) + 12r + 2 = 20$

$\Rightarrow 7r = -6 + 20 = 14 \ \therefore \ r = 2$

Hence, the required point is (8, 7, 26)

11. (a, c) : Let $\vec{r}$ be a vector in the plane of $\vec{b}$ and $\vec{c}$.

$\therefore \vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} + \hat{j} - 2\hat{k})$

$= (1 + \lambda)\hat{i} + (2 + \lambda)\hat{j} - (1 + 2\lambda)\hat{k}$

Now, the projection of $\vec{r}$ on $\vec{a}$ $= \frac{\vec{a} \cdot \vec{r}}{|\vec{a}|}$

$$= \frac{\{(1+\lambda)\hat{i} + (2+\lambda)\hat{j} - (1+2\lambda)\hat{k}\} \cdot (2\hat{i} - \hat{j} + \hat{k})}{\sqrt{2^2 + 1^2 + 1^2}}$$

$$= \frac{2(1+\lambda) - (2+\lambda) - (1+2\lambda)}{\sqrt{6}} = -\frac{(\lambda+1)}{\sqrt{6}}$$

Now, $-\frac{(\lambda+1)}{\sqrt{6}} = \pm\sqrt{\frac{2}{3}} \Rightarrow -\frac{(\lambda+1)}{\sqrt{2} \cdot \sqrt{3}} = \pm\frac{\sqrt{2}}{\sqrt{3}}$

$\Rightarrow -(\lambda + 1) = \pm 2 \Rightarrow \lambda = 1, -3$

$\therefore \vec{r} = 2\hat{i} + 3\hat{j} - 3\hat{k}$ and $-2\hat{i} - \hat{j} + 5\hat{k}$

12. (a, c) : Let θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$.

Since $\vec{a}$ and $\vec{b}$ are non-collinear, therefore $\theta \neq 0$, and $\theta \neq \pi$. We have

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = \cos\theta$

Now, $|\vec{u}|^2 = |\vec{a} - (\vec{a} \cdot \vec{b})\vec{b}|^2 = |\vec{a} - \cos\theta\vec{b}|^2$

$= |\vec{a}|^2 + \cos^2\theta|\vec{b}|^2 - 2\cos\theta(\vec{a} \cdot \vec{b})$

$= 1 + \cos^2\theta - 2\cos^2\theta = 1 - \cos^2\theta$

$\Rightarrow |\vec{u}|^2 = \sin^2\theta$ ...(i)

Also, $|\vec{v}|^2 = |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$

$\Rightarrow |\vec{v}|^2 = \sin^2\theta$ ...(ii)

$\therefore$ From (i) and (ii), we have

$|\vec{v}|^2 = |\vec{u}|^2$ or $|\vec{v}| = |\vec{u}|$

Also $\vec{u} \cdot \vec{a} = (\vec{a} - (\vec{a} \cdot \vec{b})\vec{b}) \cdot \vec{a} = \vec{a} \cdot \vec{a} - (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{a})$

$\Rightarrow |\vec{u} \cdot \vec{a}| = |\vec{a}|^2 - \cos^2\theta = 1 - \cos^2\theta = \sin^2\theta$

$\therefore |\vec{u}| + |\vec{u} \cdot \vec{a}| = \sin\theta + \sin^2\theta \neq |\vec{v}|$

Now, $\vec{u} \cdot \vec{b} = (\vec{a} - (\vec{a} \cdot \vec{b})\vec{b}) \cdot \vec{b}$

$= \vec{a} \cdot \vec{b} - (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{b}) = \vec{a} \cdot \vec{b} - (\vec{a} \cdot \vec{b}) = 0$

$\therefore |\vec{u}| + |\vec{u} \cdot \vec{b}| = |\vec{u}| + 0 = |\vec{u}| = |\vec{v}|$

Also $\vec{u} \cdot (\vec{a} + \vec{b}) = \vec{u} \cdot \vec{a} + \vec{u} \cdot \vec{b} = \vec{u} \cdot \vec{a}$ [$\because \vec{u} \cdot \vec{b} = 0$]

$\therefore |\vec{u}| + \vec{u} \cdot (\vec{a} + \vec{b}) = |\vec{u}| + \vec{u} \cdot \vec{a} \neq |\vec{v}|$

13. (a) : The two straight lines $\frac{x-4}{1} = \frac{y-3}{1} = \frac{6-z}{k}$ and $\frac{x-3}{k} = \frac{y-4}{2} = \frac{z-7}{1}$ are said to be coplanar, if

$$\begin{vmatrix} -3-(-4) & -4-(-3) & -7-(-6) \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 1(1 + 2k) + 1(1 + k^2) - 1(2 - k) = 0$

$\Rightarrow k^2 + 3k = 0 \Rightarrow k(k + 3) = 0 \Rightarrow k = 0, -3$

14. (a, b, c, d) : The given plane is $7x - 2y + z + 21 = 0$

Now $7(-1) - 2(7) + 0 + 21 = 0$

$\therefore$ The plane contains the point $(-1, 7, 0)$

Now the plane $7x - 2y + z + 21 = 0$ will contain the straight line $\frac{x+2}{1} = \frac{3-y}{-2} = \frac{z+1}{-3}$

When the normal to the plane is perpendicular to the straight line

$\therefore \quad 7(1) - 2(2) + 1(-3) = 0$

Thus the plane contains the straight line.

Also, the plane $7x - 2y + z + 21 = 0$ will be perpendicular to the straight line $\frac{x}{7} = \frac{y}{-2} = \frac{7}{1}$, when the direction ratios of the normal to the plane are same as that of the straight line. In this case, it is true.

Clearly, the plane $7x - 2y + z + 21 = 0$ and $7x - 2y + z + 5 = 0$ are parallel.

15. (a, d)

16. (d) : $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 2\hat{i} + \hat{j} - 3\hat{k}$

$$(\vec{a} \times \vec{b}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 2 & -1 & 1 \end{vmatrix} = -2\hat{i} - 8\hat{j} - 4\hat{k}$$

which is along, $\hat{i} + 4\hat{j} + 2\hat{k}$.

17. (c) : The Rolle's theorem is applicable in $[\alpha, \beta]$ if $f(\alpha) = f(\beta) = \lambda$

$\Rightarrow$ For two different values of x, the value of $f(x)$ is same, so $f(x)$ is many-one function.

$\therefore$ Statement-1 is true.

Again for L.M.V theorem

$$f'(c) = \frac{f(\alpha) - f(\beta)}{\alpha - \beta} \forall\, c \in (\alpha, \beta)$$

and $f'(c) \neq 0 \Rightarrow f(\alpha) \neq f(\beta)$

$\Rightarrow$ $f(x)$ is 1 – 1 function. If $f(\alpha) = f(\beta)$, then Rolle's theorem is special case of L.M.V.

$\therefore$ Statement-2 is false.

18. (b) : $f(x) = (x - 5)^3 \Rightarrow f'(x) = 3(x - 5)^2$

$f'(x) = 0$ for maximum or minimum $\Rightarrow x = 5$

$f''(x) = 6(x - 5)$ and $f'''(x) = 6$

$\therefore$ $f''(5) = 0$ but $f'''(5) \neq 0$

$\Rightarrow$ $f(x)$ has neither maximum nor minimum value at $x = 5$.

19. (b) : We have, $f(x) = (x - 2)(x - 3)(x - 4)$

$= x^3 - 9x^2 + 26x - 24$

$\Rightarrow f'(x) = 3x^2 - 18x + 26$

Put $f'(x) = 0$ for tangent parallel to x-axis

$\Rightarrow 3x^2 - 18x + 26 = 0$

$\Rightarrow x = \frac{9 \pm \sqrt{3}}{3} = 3 \pm \frac{1}{\sqrt{3}} \Rightarrow 2 \leq 3 \pm \frac{1}{\sqrt{3}} \leq 4$

$\Rightarrow$ Statement-1 & Statement-2 both are correct as $f(2) = f(4) = 0$ and $f'(c) = 0$ gives $c = 3 \pm \frac{1}{\sqrt{3}} \in (2, 4)$, but these conditions are not given in Statement-2.

20. (d) : Let θ be the angle between the normals $\vec{n}_1 = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{n}_2 = \hat{i} + \hat{j} + 2\hat{k}$

$$\therefore \quad \cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{2(1) + (-1)(1) + 1(2)}{\sqrt{6} \cdot \sqrt{6}} = \frac{1}{2}$$

$\Rightarrow \theta = 60°$

21. (0) : Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$\therefore$ $P'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$P'(1) = 0, \; P'(2) = 0$

Now, $\lim_{x \to 0} \left(1 + \frac{P(x)}{x^2}\right) = 2 \Rightarrow \lim_{x \to 0} \left(\frac{x^2 + P(x)}{x^2}\right) = 2$

So, $P(0) = 0 \Rightarrow e = 0$

Again $\lim_{x \to 0} \left(\frac{2x + P'(x)}{2x}\right) = 2$

$\therefore$ $P'(0) = 0 \Rightarrow d = 0$

Again $\lim_{x \to 0} \left(\frac{2 + P''(x)}{2}\right) = 2$

$\therefore$ $P''(0) = 2 \Rightarrow 2c = 2 \;\; \therefore c = 1$

Since, $P(2) - P'(2) = 0$ gives $a = \frac{1}{4}, b = -1$

$$\therefore \quad P(x) = \frac{x^4}{4} - x^3 + x^2$$

Hence $P(2) = 0$

22. (7) : As $A = \{x : x^2 + 20 \leq 9x\}$

$= \{x : x^2 - 9x + 20 \leq 0\}$

$= \{x : (x - 4)(x - 5) \leq 0\}$

We have, $A = [4, 5]$

Now $f(x) = 2x^3 - 15x^2 + 36x - 48$

$\Rightarrow f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$= 6(x - 2)(x - 3)$

Thus f has no critical point in$[4, 5]$ as $f'(0) \neq 0$ in $(4, 5)$ and $f'(x)$ exists at all points.

$f(4) = 2\cdot4^3 - 15\cdot4^2 + 36\cdot4 - 48 = -16$

$f(5) = 2\cdot5^3 - 15\cdot5^2 + 36\cdot5 - 48 = 7$

Thus maximum value of f on $[4, 5]$ is 7.

23. (6) : Equation of the plane containing the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0 \Rightarrow x - 2y + z = 0$$

Comparing with $Ax - 2y + z = d$, we have $A = 1$

$\therefore$ The distance between the planes

$$= \frac{d}{\sqrt{1^2+2^2+1^2}} = \frac{d}{\sqrt{6}} \quad \therefore \quad \frac{d}{\sqrt{6}} = \sqrt{6} \Rightarrow |d| = 6$$

24. (5) : Here $\vec{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$

$$\therefore \quad |\vec{a}| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5}+\frac{4}{5}} = 1$$

and $\vec{b} = \frac{2\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{14}} = \frac{2}{\sqrt{14}}\hat{i} + \frac{1}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$

$$\therefore \quad |\vec{b}| = \sqrt{\left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2} = 1$$

Now, $\vec{a}\cdot\vec{b} = \frac{(\hat{i} - 2\hat{j})\cdot(2\hat{i} + \hat{j} + 3\hat{k})}{\sqrt{5}\times\sqrt{14}} = 0$

Also, $(2\vec{a} + \vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a} - 2\vec{b})]$

$= (2\vec{a} + \vec{b})\cdot[(\vec{a}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{a} - 2(\vec{b}\cdot\vec{a})\vec{b} + 2(\vec{b}\cdot\vec{b})\vec{a}]$

$= (2\vec{a} + \vec{b})\cdot[|\vec{a}|^2\,\vec{b} + 2|\vec{b}|^2\,\vec{a}]$

$= (2\vec{a} + \vec{b})\cdot(\vec{b} + 2\vec{a}) = (2\vec{a} + \vec{b})^2 \qquad (\because |\vec{a}| = 1, |\vec{b}| = 1)$

$= 4|\vec{a}|^2 + |\vec{b}|^2 = 4 + 1 = 5$

25. (2) : $y = -2x + 1$ passes through nearest directrix

$$\Rightarrow \quad 0 = -\frac{2a}{e} + 1 \Rightarrow \frac{2a}{e} = 1 \Rightarrow \frac{a}{e} = \frac{1}{2} \Rightarrow e = 2a$$

Since $y = -2x + 1$ is tangent to hyperbola

$\therefore \quad 1 = 4a^2 - b^2 \Rightarrow b^2 = 4a^2 - 1$

Now, $e^2 = 1 + \frac{b^2}{a^2} = 5 - \frac{4}{e^2}$

$\Rightarrow \quad e^4 - 5e^2 + 4 = 0 \Rightarrow (e^2 - 1)(e^2 - 4) = 0$

$\therefore \quad e^2 = 4 \Rightarrow e = 2$

PAPER-II

1. (d) : (a) $f(x) = |x|[x] = \begin{cases} 0, & \text{when } 0 \le x < 1 \\ x, & \text{when } -1 \le x < 0 \end{cases}$

So, $f(x)$ is not differentiable at $x = 0$

(b) $f(x) = [x] + [-x] = \begin{cases} 0, & \text{when } x \in I \\ -1, & \text{when } x \notin I \end{cases}$

and $f(0) = 0$, so $f(x)$ is not continuous at $x = 0$

(c) $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0} \frac{\sin x}{x} = 1$ and $f(0) = 0$

So $f(x)$ is not continuous at $x = 0$

(d) $f(x) = |x| - |\sin x|, f(-1) = f(1)$

and $f(x) = \begin{cases} x - \sin x, & 0 \le x \le 1 \\ -x + \sin x, & -1 \le x < 0 \end{cases}$

$\therefore$ $f(x)$ is both continuous and differentiable at $x = 0$.

2. (a)

3. (d) : We know that equation of bisector of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, where position vector of A, B and C are respectively $\vec{a}, \vec{b}$ and $\vec{c}$ is $\vec{r} = \vec{a} + t(\hat{b} \pm \hat{c})$.

Here A is origin, thus the required equation is

$$\vec{r} = t\left\{\left(\frac{\hat{i} + 3\hat{j} - 2\hat{k}}{\sqrt{14}}\right) \pm \left(\frac{3\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{14}}\right)\right\}$$

$$\Rightarrow \quad \vec{r} = \frac{t}{\sqrt{14}}(-2\hat{i} + 2\hat{j}) \text{ or } \vec{r} = \frac{t}{\sqrt{14}}(4\hat{i} + 4\hat{j} - 4\hat{k})$$

At $t = \frac{\sqrt{14}}{4}$, the second bisector is $\hat{i} + \hat{j} - \hat{k}$.

4. (c)

5. (d) : The shortest distance between $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ is

$$\frac{|(\vec{c} - \vec{a})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$$

Here $\vec{a} = -(\hat{i} + \hat{j} + \hat{k}), \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{c} = -\hat{i}, \ \vec{d} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Now, $\vec{c}-\vec{a}=-\hat{i}+\hat{i}+\hat{j}+\hat{k}=\hat{j}+\hat{k}$

$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&3&4\\3&4&5\end{vmatrix}=-\hat{i}+2\hat{j}-\hat{k}$$

and $|\vec{b}\times\vec{d}|=\sqrt{1+4+1}=\sqrt{6}$

$\therefore$ The required distance is

$$\frac{(0\cdot\hat{i}+\hat{j}+\hat{k})\cdot(-\hat{i}+2\hat{j}-\hat{k})}{\sqrt{6}}=\frac{0+2-1}{\sqrt{6}}=\frac{1}{\sqrt{6}}$$

6. (d): Let $f(x) = ax^3 + bx^2 + cx + d$

Since $f'(x)$ has minimum at $x = 1$

$\therefore$ $f''(1) = 0 \Rightarrow 6a + 2b = 0 \Rightarrow 3a + b = 0$

Again $f(x)$ has maximum at $x = -1$

$\Rightarrow$ $f'(-1) = 0 \Rightarrow 3a - 2b + c = 0$

$\Rightarrow$ $3a + 6a + c = 0 \Rightarrow 9a + c = 0 \Rightarrow c = -9a$

$\therefore$ $f'(x) = 3ax^2 - 6ax - 9a = 3a(x^2 - 2x - 3)$

$= 3a(x + 1)(x - 3)$

$\therefore$ $f(x)$ has minimum at $x = 3$.

7. (d): Let $x = 3t^2$ and $y = 2t^3$ $\therefore$ $\frac{dy}{dx}=t$

Equation of the tangent at t is

$y - 2t^3 = t(x - 3t^2) \Rightarrow tx - y = t^3$ ...(i)

The normal at t_1 is

$$y-2t_1^3=-\frac{1}{t_1}(x-3t_1^2)$$

$\Rightarrow$ $t_1y + x = 2t_1^4 + 3t_1^2$ ...(ii)

$\because$ (i) & (ii) are identical.

$\therefore$ $-t^3=2t_1^3+3t_1$ and $t_1=-\frac{1}{t}$

Therefore, we get $t^2 = 2$, $t=\pm\sqrt{2}$

The lines are $y=\pm\sqrt{2}(x-2)$

8. (c): $g(x)=2f\left(\frac{x}{2}\right)+f(2-x)$

$$\Rightarrow\quad g'(x)=f'\left(\frac{x}{2}\right)-f'(2-x)$$

Now $g'(x)>0$ if $f'\left(\frac{x}{2}\right)>f'(2-x)\Rightarrow \frac{x}{2}<2-x$

[$\because$ $f'(x)$ is a decreasing function, as $f''(x) < 0$]

$\Rightarrow$ $3x<4 \Rightarrow x<\frac{4}{3}$

Thus, if $x\in\left(0,\frac{4}{3}\right)$, $g(x)$ increases.

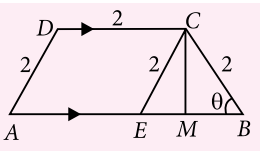
9. (c): Here $BC = CD = DA = 2$.

Draw CE parallel to DA.

$\angle ABC = \theta$

$\Rightarrow$ $EB = 4\cos\theta$,

$AB = 2 + 4\cos\theta$



$\therefore$ Area $A=\frac{1}{2}(AB+DC)CM$

$$=\frac{1}{2}(2+4\cos\theta+2)2\sin\theta=4(1+\cos\theta)\sin\theta$$

$\Rightarrow$ $A = 4\sin\theta + 2\sin2\theta$

$\frac{dA}{d\theta}=0 \Rightarrow 4\cos\theta+4\cos2\theta=0 \Rightarrow \theta=\frac{\pi}{3}$

Now, $\frac{d^2A}{d\theta^2}=-(4\sin\theta+8\sin2\theta)$, $\left.\frac{d^2A}{d\theta^2}\right|_{\theta=\frac{\pi}{3}}<0$

$\therefore$ The area is maximum at $\theta=\frac{\pi}{3}$.

$\therefore$ $A_{max}=4\left(1+\frac{1}{2}\right)\frac{\sqrt{3}}{2}=3\sqrt{3}$.

10. (d): The equation of PQ is

$$\frac{x-1}{1}=\frac{y+3}{3}=\frac{z-2}{2}$$

Any point on PQ has coordinates $(1 + r, -3 + 3r, 2 + 2r)$

Since Q is a point on the plane $x + y - 3z = 0$

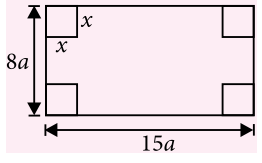
$\therefore$ $1 + r - 3 + 3r - 3(2 + 2r) = 0 \Rightarrow r = -4$

$\therefore$ $Q \equiv (-3, -15, -6)$

11. (a, c): The volume is given by

$V = (15a - 2x)(8a - 2x)x$

And it is given that volume V is maximum at $x = 5$ *i.e.*, $2x = 10$. We have,



$V(t) = (15a - t)(8a - t)\frac{t}{2}$

$$=\frac{1}{2}(t^3-23at^2+120a^2t)$$

$\therefore$ $\frac{dV}{dt}=\frac{1}{2}(3t^2-46at+120a^2)$

Now $\left.\frac{dV}{dt}\right|_{t=10}=0 \Rightarrow 120a^2 - 460a + 300 = 0$

$\Rightarrow$ $6a^2 - 23a + 15 = 0 \Rightarrow (6a - 5)(a - 3) = 0$

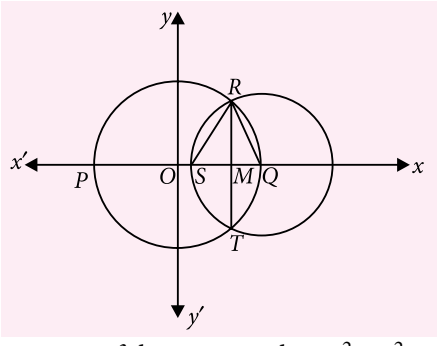
$\therefore$ $a=3,\frac{5}{6}$

Also, $\frac{d^2V}{dt^2}=\frac{1}{2}\{6t-46a\}=3t-23a$

Since $\left.\frac{d^2V}{dt^2}\right|_{t=10,\ a=3} < 0$

$\therefore$ V is maximum at $a = 3$. Thus the sides are 24 and 45.

12. (a, b, d) :



The equation of the given circle is $x^2 + y^2 = 1$ ...(i)

It cuts the x-axis at $P(-1, 0)$ and $Q(1, 0)$

Equation of a circle with $Q(1, 0)$ as centre and radius r is $(x - 1)^2 + (y - 0)^2 = r^2$, r being the variable

i.e. $x^2 + y^2 - 2x + (1 - r^2) = 0$, $(0 < r < 2)$ ...(ii)

Solving (i) and (ii), we get

$$x = \frac{2 - r^2}{2},\ y = \pm\frac{r\sqrt{4 - r^2}}{2}$$

But R is above the x-axis. $\therefore$ $R\left(\frac{2 - r^2}{2}, \frac{r\sqrt{4 - r^2}}{2}\right)$

Now $SQ = r$, MR = ordinate of $R = \frac{r\sqrt{4 - r^2}}{2}$

Area of $\Delta QRS = \frac{1}{2} r \cdot \frac{r\sqrt{4 - r^2}}{2}$

$\Rightarrow$ $\Delta = \frac{r^2\sqrt{4 - r^2}}{4}$, $\Delta^2 = \frac{r^4(4 - r^2)}{16} = A$ (say)

$\frac{dA}{dr} = \frac{1}{16}(16r^3 - 6r^5)$, $\frac{d^2A}{dr^2} = \frac{1}{16}(48r^2 - 30r^4)$

For maximum or minimum $\frac{dA}{dr} = 0 \Rightarrow r^2 = \frac{8}{3}$

$$\left.\frac{d^2A}{dr^2}\right|_{r=\sqrt{\frac{8}{3}}} = \frac{1}{16}\left(48 \times \frac{8}{3} - 30 \times \frac{64}{9}\right) < 0$$

$\therefore$ A is maximum and Δ is maximum at $r = \sqrt{\frac{8}{3}}$.

$$\Delta_{max.} = \frac{1}{2} \times \frac{8}{3} \frac{\sqrt{4 - \frac{8}{3}}}{2} = \frac{4\sqrt{3}}{9}$$

13. (b, c) : Let $f(x) = ax^3 + bx^2 + cx + d$

$\because$ $f(2) = 18$ $\therefore$ $8a + 4b + 2c + d = 18$ ...(i)

Again, $f(1) = -1$ $\therefore$ $a + b + c + d = -1$ ...(ii)

and $f(x)$ has local maximum at $x = -1$

$\therefore$ $f'(-1) = 0 \Rightarrow 3a - 2b + c = 0$ ...(iii)

Moreover $f'(x)$ has local minimum at $x = 0$

$\Rightarrow$ $f''(0) = 0 \Rightarrow b = 0$ ...(iv)

Solving (i), (ii), (iii) and (iv), we get

$$f(x) = \frac{1}{4}(19x^3 - 57x + 34) \Rightarrow f(0) = \frac{17}{2}$$

Also, $f'(x) = \frac{57}{4}(x^2 - 1) > 0,\ \forall\ x > 1$

Also $f'(x) = 0 \Rightarrow x = 1, -1$

$f''(-1) < 0$, $f''(1) > 0 \Rightarrow x = -1$ is a point of local maximum and $x = 1$ is a point of local minimum.

$f(x)$ is increasing $\Rightarrow$ $f'(x) \geq 0 \Rightarrow x^2 - 1 \geq 0$

$\Rightarrow$ $x \leq -1$ or $x \geq 1$.

$\therefore$ Distance between $(-1, 2)$ and $(1, f(1))$ *i.e.* $(1, -1)$ is $\sqrt{13}$.

14. (b, d) : Let $y = ax^2 + bx + c$. As it touches $y = x$ at $x = 1$ $\therefore$ $y = a + b + c$ and $y = 1 \Rightarrow a + b + c = 1$

Also, $\frac{dy}{dx} = 2ax + b$

$\therefore$ $\left.\frac{dy}{dx}\right|_{x=1} = 2a + b = 1$ ($\because$ from $y = x$, the slope $= 1$)

Now, $f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$ and $f''(x) = 2a$

$\therefore$ $f(0) = c, f'(0) = b, f''(0) = 2a, f'(1) = 2a + b = 1$

But $a + b + c = 1 \Rightarrow 2a + 2b + 2c = 2$

$\Rightarrow$ $1 + b + 2c = 2 \Rightarrow 2c = 1 - b$

15. (a, c)

16. (c) : $L_1 : 2x - 2y + 3z - 2 = 0 = x - y + z + 1$

$L_2 : x + 2y - z - 3 = 0 = 3x - y + 2z - 1$

The above two equations in symmetric form are

$$\frac{x + 5}{1} = \frac{y}{1} = \frac{z - 4}{0} \text{ and } \frac{x - \frac{5}{7}}{3} = \frac{y - \frac{8}{7}}{-5} = \frac{z}{-7}$$

$\therefore$ Equation of the plane through L_1 and L_2 is

$$\begin{vmatrix} x + 5 & y & z - 4 \\ 1 & 1 & 0 \\ 3 & -5 & -7 \end{vmatrix} = 7x - 7y + 8z + 3 = 0 \quad ...(i)$$

$\therefore$ The distance of plane (i) from the origin

$$= \frac{3}{\sqrt{162}} = \frac{1}{3\sqrt{2}}.$$

17. (a) : The general points lying on two lines L_1 and L_2 is given by

$(\lambda-5, \lambda, 4)$ and $\left(3\mu+\frac{5}{7}, -5\mu+\frac{8}{7}, -7\mu\right)$

respectively, where λ and μ are real parameters. If the above two straight lines intersect, then for some λ and μ we must have,

$$\lambda-5=3\mu+\frac{5}{7} \quad ...(i), \qquad \lambda=-5\mu+\frac{8}{7} \quad ...(ii)$$

$$4=-7\mu \quad ...(iii)$$

From (iii), we get $\mu=-\frac{4}{7}$

Putting $\mu=-\frac{4}{7}$ in (ii), we get $\lambda=4$

Putting these values of λ and μ in (i), we get

$$\text{L.H.S}=-1,\ \text{R.H.S}=3\left(-\frac{4}{7}\right)+\frac{5}{7}=-1$$

Therefore two lines intersect each other.

Hence the point of intersection is $P(-1, 4, 4)$

$\therefore \quad OP=\sqrt{33}$

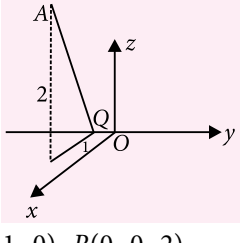
18. (d) **19. (c)**

20. (a) : $AQ^2 = 1^2 + 2^2 = 5$

Similarly $AP^2 = (-1)^2 + 2^2 = 5$

and $AR^2 = 1^2 + (-1)^2 = 2$

$\therefore$ A.M. $=\frac{5+5+2}{3}=4$



21. (b) : $P \equiv (1, 0, 0)$, $Q(0, -1, 0)$, $R(0, 0, 2)$

Any plane through P is

$a(x-1)+b(y-0)+c(z-0)=0$

It passes through Q

$\Rightarrow \quad a(0-1)+b(-1-0)+c(0-0)=0$

It passes through R

$\Rightarrow \quad a(0-1)+b(0-0)+c(2-0)=0$

$\therefore$ The equation of the plane PQR is

$$\begin{vmatrix} x-1 & y & z \\ -1 & -1 & 0 \\ -1 & 0 & 2 \end{vmatrix}=0$$

$\Rightarrow \quad -2(x-1)-y(-2)+z(0-1)=0$

$\Rightarrow \quad 2x-2y+z=2$

22. (b) : $(\vec{\alpha}\times\vec{\beta})\cdot\vec{\gamma}=\vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma})$

$=(\vec{b}\times\vec{c})\cdot[(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$

$=(\vec{b}\times\vec{c})\cdot[((\vec{c}\times\vec{a})\cdot\vec{b})\,\vec{a}-((\vec{c}\times\vec{a})\cdot\vec{a})\vec{b}]$

$=(\vec{b}\times\vec{c})\cdot[\vec{c}\ \vec{a}\ \vec{b}]\,\vec{a}=[\vec{b}\ \vec{c}\ \vec{a}][\vec{c}\ \vec{a}\ \vec{b}]$

$=[\vec{c}\ \vec{a}\ \vec{b}][\vec{c}\ \vec{a}\ \vec{b}]=[\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}]$

$\therefore \quad (\vec{\alpha}\times\vec{\beta})\cdot\vec{\gamma}=[\vec{a}\ \vec{b}\ \vec{c}]^2$

23. (c) : $[\vec{\beta}\times\vec{\gamma}\ \ \vec{\gamma}\times\vec{\alpha}\ \ \vec{\alpha}\times\vec{\beta}]$

$=(\vec{\beta}\times\vec{\gamma})\cdot((\vec{\gamma}\times\vec{\alpha})\times(\vec{\alpha}\times\vec{\beta}))$

$=(\vec{\beta}\times\vec{\gamma})\cdot\{((\vec{\gamma}\times\vec{\alpha})\cdot\vec{\beta})\vec{\alpha}-((\vec{\gamma}\times\vec{\alpha})\cdot\vec{\alpha})\vec{\beta}\}$

$=(\vec{\beta}\times\vec{\gamma})\cdot[\vec{\gamma}\ \vec{\alpha}\ \vec{\beta}]\vec{\alpha}$

$=[\vec{\beta}\ \vec{\gamma}\ \vec{\alpha}][\vec{\gamma}\ \vec{\alpha}\ \vec{\beta}]=[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}][\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]$

$=[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]^2=\{[\vec{a}\ \vec{b}\ \vec{c}]^2\}^2=[\vec{a}\ \vec{b}\ \vec{c}]^4$

24. A → r, B → s, C → p, D → q

(A) Volume of a parallelopiped determined by vectors $\vec{a}, \vec{b}, \vec{c}$ is 2.

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}]=2$

$\therefore \quad [2(\vec{a}\times\vec{b})\,3(\vec{b}\times\vec{c})\,(\vec{c}\times\vec{a})]=6[\vec{a}\ \vec{b}\ \vec{c}]^2=6\times4=24$

(B) $[\vec{a}\ \vec{b}\ \vec{c}]=5$

$\therefore \quad [3(\vec{a}+\vec{b})\ (\vec{b}+\vec{c})\ 2(\vec{c}+\vec{a})]$

$=3(\vec{a}+\vec{b})\cdot((\vec{b}+\vec{c})\times2(\vec{c}+\vec{a}))$

$=6(\vec{a}+\vec{b})\cdot((\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{c})+(\vec{c}\times\vec{a}))$

$=6(\vec{a}+\vec{b})\cdot((\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a}))$

$=6(\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})$

$\qquad +\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a}))$

$=6\{[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]\}=6\{[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{c}]\}$

$=6\times2[\vec{a}\ \vec{b}\ \vec{c}]=12[\vec{a}\ \vec{b}\ \vec{c}]=12\times5=60$

(C) Since area of triangle $=\frac{1}{2}|\vec{a}\times\vec{b}|=20$

$\therefore \quad \frac{1}{2}|(2\vec{a}+3\vec{b})\times(\vec{a}-\vec{b})|=\frac{1}{2}|3(\vec{b}\times\vec{a})-2(\vec{a}\times\vec{b})|$

$=\frac{1}{2}\left|-3(\vec{a}\times\vec{b})-2(\vec{a}\times\vec{b})\right|=\frac{1}{2}\cdot5|\vec{a}\times\vec{b}|$

$=5\times20=100$

(D) $|\vec{a}\times\vec{b}|=30$

$\therefore \quad |(\vec{a}+\vec{b})\times\vec{a}|=|\vec{b}\times\vec{a}|=|-\vec{a}\times\vec{b}|=|\vec{a}\times\vec{b}|=30$

25. A → r; B → p; C → q; D → s

PRACTICE PAPER 2015

JEE (Main & Advanced) and other PETs

PAPER-1

SECTION-I

MULTIPLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct.

1. The function $f(x)=\begin{cases}|x-3|, x\geq 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, x<1\end{cases}$ is

(a) continuous at $x=1$ (b) derivable at $x=1$

(c) not derivable at $x=1$ (d) not derivable at $x=3$

2. The solutions of the quadratic equation $(3|x|-3)^2=|x|+7$, which belongs to the domain of definition of the function $y=\sqrt{(x-4)x}$ is/are

(a) $\frac{1}{9}$ (b) 2 (c) -2 (d) $-\frac{1}{9}$

3. Which of the following functions (is) are injective maps?

(a) $f(x)=|x+1|, x\in[-1,\infty)$

(b) $g(x)=x+\frac{1}{x}, x\in(0,\infty)$

(c) $h(x)=x^2+4x-5, x\in(0,\infty)$

(d) $k(x)=e^{-x}, x\in[0,\infty)$

4. Let $y=g(x)=\frac{x+2}{x-1}$, then

(a) $g(1)=3$ (b) $x=g(y)$

(c) y increases with x for $x>1$

(d) g is a rational function of x

5. For all $x\in(0,1)$

(a) $e^x<1+x$ (b) $\log_e(1+x)<x$

(c) $\sin x>x$ (d) $\log_e x>x$

6. A function $f: R\to R$ satisfies the equation $f(x+y)=f(x)f(y)$ for all $x, y\in R$ and $f(x)\neq 0$ for any x in R. Let function be differentiable at $x=0$ and $f'(0)=2$. Then

(a) $f(x)$ is continuous at $x=0$.

(b) $f(x)$ is not differentiable at infinitely many points.

(c) $f(x)=e^{2x}$ (d) None of these

7. The function $f(x)=1+|\sin x|$ is

(a) continuous everywhere.

(b) not differentiable at an infinite number of points.

(c) differentiable nowhere.

(d) not differentiable at $x=0$.

8. Let $[x]$ denote the greatest integer less than or equal to x. If $f(x)=[x\sin\pi x]$, then $f(x)$ is

(a) continuous at $x=0$

(b) continuous in $(-1,0)$

(c) differentiable at $x=1$

(d) differentiable in $(-1,1)$

9. If $f(x)=1+2\sin x+2\cos^2 x, 0\leq x\leq \pi/2$, then $f(x)$ is

(a) greatest at $x=\pi/6$ (b) least at $x=\pi/6$

(c) increasing in $\left(0,\frac{\pi}{6}\right)$ and decreasing in $\left(\frac{\pi}{6},\frac{\pi}{2}\right)$

(d) least at $x=0$ and $x=\pi/2$

10. If $f(x)=\sin^3 x+\lambda\sin^2 x, -\frac{\pi}{2}<x<\frac{\pi}{2}$, then

(a) $f(x)$ has a point of inflexion if $\lambda=0$

(b) $f(x)$ has exactly one point of maximum and exactly one point of minimum if $|\lambda|<3/2$

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(c) $f(x)$ has exactly one point of maximum and exactly one point of minimum if

$$\lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$$

(d) all of the above

SECTION-II

ONE INTEGER VALUE CORRECT TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive).

11. $\lim_{x\to 0}\left[\frac{\sin x}{x}\right]$, where $[\cdot]$ denotes the greatest integer function is

12. Consider the function $f(x) = x - |x - x^2|$, $-1 \le x \le 2$. Find the number of points of non-differentiability of $f(x)$ for $x \in [-1, 2]$

13. $\lim_{x\to 0} \frac{\cos^2(1-\cos^2(1-\cos^2(....-\cos^2(x))))....)}{\sin\left\{\pi\left(\frac{\sqrt{x+4}-2}{x}\right)\right\}} = \frac{k}{\pi}$,

where $k =$

14. The left hand limit of $f(x) = \left\{\frac{|x|^3}{3} - \left[\frac{x}{3}\right]^3\right\}$, where $[x]$ denotes the greatest integer less than or equal to x, is

15. The derivative of $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$ is at $x = \frac{\pi}{3}$ is $k - 5$, if $k =$

16. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

17. If $f(x + y) = f(x) \cdot f(y)$ for all x and y and $f(5) = 2$, $f'(0) = 3$, then $f'(5)$ is equal to

18. A food company produces x quality A and y quality B items per day, where $y(5 - x) = 10(4 - x)$, $0 \le x \le 4$. If the profit on each quality A item is twice the profit on quality B item, then the most profitable number of quality A items per day to manufacture is

19. If $f(x) = \begin{cases} 2 - |x^2 + 5x + 6|, x \ne -2 \\ a^2 + 1, x = -2 \end{cases}$,

then the minimum value of $|a|$ so that $f(x)$ has maxima at $x = -2$ is

20. From a fixed point A on the circumference of a circle of radius r, the perpendicular AY is let fall on the tangent at P. The maximum area of the ΔAPY is $\frac{3\sqrt{3}}{k}r^2$, for $k =$

PAPER-2

SECTION-I

SINGLE CORRECT OPTION

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

1. $\lim_{x\to 1}\frac{x^{n+1} - (n+1)x + n}{(x-1)^2} =$

(a) $n(n + 1)$ (b) $\frac{n(n+1)}{2}$

(c) $n + 1$ (d) $\frac{3n}{2}$

2. $\lim_{x\to 0}\frac{\tan\sqrt[3]{x}\ln(1+3x)}{(\tan^{-1}\sqrt{x})^2 e^{5\sqrt[3]{x}} - 1} =$

(a) $\frac{\sqrt{3}}{5}$ (b) 1

(c) $\frac{3}{5}$ (d) 0

3. Value of $\lim_{n\to\infty}\left\{\prod_{i=1}^{n}\cos\left(\frac{x}{2^i}\right)\right\}$ is

(a) $\sin x$ (b) $(\sin x)/x$

(c) $x/\sin x$ (d) 1

4. $\lim_{x\to 0}(\text{cosec}^2 x + 2^{\text{cosec}^2 x} + ... + n^{\text{cosec}^2 x})\sin^2 x$

(a) n (b) n^2 (c) 0 (d) 1

5. If $y = \sin x$, then $\frac{d^2\cos^7 x}{dy^2}$ is equal to

(a) $35\cos^3 x - 42\cos^5 x$ (b) $35\cos^3 x + 42\cos^5 x$

(c) $42\cos^3 x - 35\cos^5 x$ (d) $-35\cos^3 x - 42\cos^5 x$

6. If $5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$ and $y = x f(x)$, then $\left.\frac{dy}{dx}\right|_{x=1}$ is equal to

(a) 14 (b) -1 (c) 1 (d) 7/8

7. If $f(x)=\frac{x}{\sin x}$ and $g(x)=\frac{x}{\tan x}$ where $0<x\le 1$, then in this interval

(a) both $f(x)$ and $g(x)$ are increasing functions

(b) both $f(x)$ and $g(x)$ are decreasing functions

(c) $f(x)$ is an increasing function

(d) None of these

8. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. The maximum possible volume of the box is given by (in cm^2)

(a) 420 (b) 338

(c) 432 (d) None of these

9. The function $f(x)=\frac{x}{1+x\tan x}$ has

(a) one point of minimum in the interval $(0, \pi/2)$

(b) one point of maximum in the interval $(0, \pi/2)$

(c) no point of maximum, no point of minimum in the interval $(0, \pi/2)$

(d) two points of maxima in the interval $(0, \pi/2)$.

10. $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$, given that $f'(2) = 6$ and $f'(1) = 4$

(a) does not exist (b) is equal to –3/2

(c) is equal to 3/2 (d) is equal to 3

SECTION-II

PARAGRAPH TYPE

This section contains 3 paragraphs. Based upon each of the paragraphs 3 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

Paragraph for Q. No. 11 and 12

In calculus, the derivative of any function $y = f(x)$ is defined as $Df(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Now instead, of this usual definition of derivative $Df(x)$, define a new kind of derivative $D^*f(x)$, which can be calculated by the formula

$D^*f(x)=\lim_{h\to 0}\frac{f^2(x+h)-f^2(x)}{h}$, where $f^2(x)=(f(x))^2$.

11. If $f(x)=\frac{x}{\ln x}$; then $D^*f(x)$ is

(a) $\frac{\ln x-1}{(\ln x)^2}$ (b) $\frac{2x(\ln x-1)}{(\ln x)}$

(c) $\frac{2x(\ln x-1)}{(\ln x)^2}$ (d) $\frac{2x(\ln x-1)}{(\ln x)^3}$

12. If function $g(x) = x^x$, then $D^*g(x)|_{x=1}$ is

(a) 1 (b) $2e^e$

(c) 2 (d) Not defined

Paragraph for Q. No. 13 and 14

Given a function 'g' which has a derivative $g'(x)$ for every real 'x' and which satisfy $g'(0) = 2$ and $g(x + y) = e^y g(x) + e^x g(y)$ for all x and y.

13. The function $g(x)$ is

(a) $x(2 + xe^x)$ (b) $x(e^x + 1)$

(c) $2x \cdot e^x$ (d) $x + \ln(x + 1)$

14. The range of the function $g(x)$ is

(a) R (b) $\left[-\frac{2}{e}, \infty\right)$

(c) $\left[-\frac{1}{e}, \infty\right)$ (d) $[0, \infty)$

Paragraph for Q. No. 15 and 16

Consider a function defined in $[-2, 2]$ is

$f(x) = \{x\}$, $-2 \le x < -1$

$= |\text{sgn}(x)|$, $-1 \le x \le 1$

$= \{-x\}$, $1 < x \le 2$, where $\{x\}$ denotes the fractional part function

15. The function $f(x)$ is continuous at

(a) $x = -2$ (b) $x = -3/2$

(c) $x = -1$ (d) All of these

16. The total number of points of discontinuity of $f(x)$ in $x \in [-2, 2]$ are

(a) 1 (b) 3 (c) 4 (d) infinite

SECTION-III

MATCHING LIST TYPE (ONLY ONE OPTION CORRECT)

This section contains 4 questions, each having two matching lists. Choices for the correct combination of elements from List-I and List-II are given as options (a), (b), (c) and (d), out of which ONE is correct.

17. Match the following:

Column I		Column II	
(P)	Period of the function $f(x) = \lvert\sin(\cos x)\rvert + \cos(\sin x)$	1.	$\pi/2$
(Q)	Period of the function $f(x) = [\sin(4x)] + \lvert\cos 4x\rvert$ ($[\cdot]$ denotes the greatest integer function)	2.	can't be determined

(R)	If the function $f : R \to R$ be such that $f(x) = \pi x - [\pi x]$, where $[\cdot]$ denotes the greatest function, then the period of the function $f^{-1}(x)$	3.	2π
(S)	Period of the function $f(x) =$ min.$\{\sin x, \lvert x\rvert\}$	4.	π

	P	Q	R	S
(a)	1	2	1	1
(b)	4	1	2	3
(c)	1	2	4	3
(d)	1	4	2	3

18. If $L = \lim_{x \to 0} \dfrac{a \sin x - bx + cx^2 + x^3}{2x^2 \log(1+x) - 2x^3 + x^4}$ and is finite, then match the following :

Column I		Column II	
(P)	$a + b$	1.	0
(Q)	c	2.	6
(R)	$20\,L$	3.	12
(S)	a	4.	1.5

	P	Q	R	S
(a)	1	2	1	4
(b)	2	4	1	3
(c)	3	1	4	2
(d)	1	4	2	3

19. Match the following :

Column I		Column II	
(P)	The ratio of altitude to the radius of the cylinder of maximum value that can be inscribed in a given sphere is	1.	$\dfrac{1}{\sqrt{2}}$
(Q)	The ratio of radius to the altitude of the cone of the greatest volume which can be inscribed in a given sphere is	2.	$\sqrt{2}$
(R)	The cone circumscribing of sphere of radius 'r' has the minimum volume if its semi vertical angle is θ is such that $33 \sin\theta =$	3.	$\dfrac{32}{3}$
(S)	The greatest value of x^3y^4 if $2x + 3y = 7$ and $x \geq 0, y \geq 0$.	4.	11

	P	Q	R	S
(a)	1	2	1	1
(b)	1	4	2	3
(c)	1	2	3	4
(d)	2	1	4	3

20. Match the following :

Column I		Column II	
(P)	Range of $\sqrt{[\sin 2x] - [\cos 2x]}$	1.	{1, 2, 3}
(Q)	Domain of $\sqrt{{}^{x^2+4^x}C_{2x^2+3}}$	2.	{1}
(R)	Range of $\sqrt{\log(\cos(\sin x))}$	3.	{0, 1}
(S)	Range of $[\lvert\sin x\rvert + \lvert\cos x\rvert]$	4.	{0, 1}

	P	Q	R	S
(a)	3	1	4	2
(b)	2	1	2	3
(c)	1	2	4	3
(d)	4	1	2	3

ANSWERS KEYS

PAPER-1

1. (a, b, d) **2.** (c, d) **3.** (a, c, d)
4. (b,d) **5.** (b) **6.** (a, c)
7. (a, b, c) **8.** (a, b, d) **9.** (a, c, d)
10. (a, c) **11.** (0) **12.** (2)
13. (4) **14.** (9) **15.** (4) **16.** (1) **17.** (6)
18. (3) **19.** (1) **20.** (8)

PAPER-2

1. (b) **2.** (c) **3.** (b) **4.** (a) **5.** (a) **6.** (d)
7. (c) **8.** (c) **9.** (b) **10.** (d) **11.** (d) **12.** (c)
13. (c) **14.** (b) **15.** (d) **16.** (b) **17.** (b) **18.** (c)
19. (d) **20.** (a)

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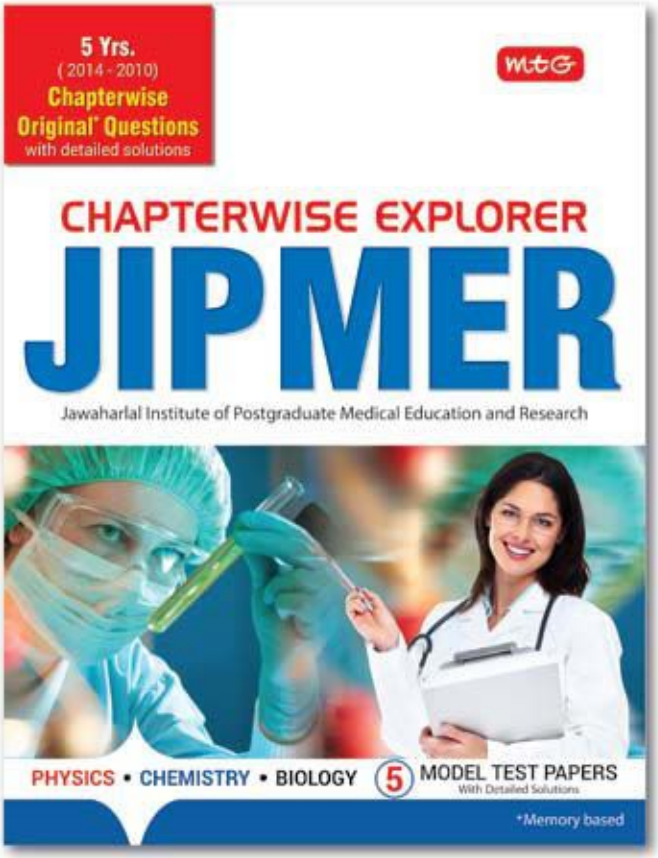

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